SKOLIAD No. 120

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **1 May, 2010**. A copy of Crux will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Maritime Mathematics Competition, 2009. Our thanks go to David Horrocks, University of Prince Edward Island, for providing us with this contest and for permission to publish it.

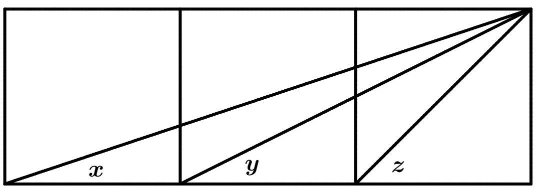
Maritime Mathematics Competition 2009
2 hours allowed

1. Two cars leave city A at the same time. The first car drives to city B at 40 km/hr and then immediately returns to city A at the same speed. The second car drives to city B at 60 km/hr and then returns to city A at a constant speed, arriving at the same time as the first car. What was the second car's speed on its return trip?

2. The perimeter of a regular hexagon H is identical to that of an equilateral triangle T. Find the ratio of the area of H to the area of T.

3. Some integers may be expressed as the sum of consecutive odd positive integers. For example, $64 = 13 + 15 + 17 + 19$. Is it possible to express 2009 as the sum of consecutive odd positive integers? If so, find all such expressions for 2009.

4. The diagram shows three squares and angles x, y, and z. Find the sum of the angles x, y, and z.



5. Suppose that x_1, x_2, x_3, x_4, and x_5 are real numbers satisfying the following equations.

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 &= 1\,, \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 &= 8\,, \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 &= 23\,. \end{aligned}$$

Find the value of $x_1 + x_2 + x_3 + x_4 + x_5$.

6. A math teacher writes the equation $x^2 - Ax + B = 0$ on the blackboard where A and B are positive integers and B has two digits. Suppose that a student erroneously copies the equation by transposing the two digits of B as well as the plus and minus signs. However, the student finds that her equation shares a root, r, with the original equation. Determine all possible values of A, B, and r.

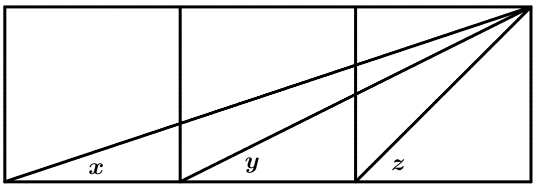
Concours de mathématiques des Maritimes 2009
Durée : 2 heures

1. Deux voitures quittent simultanément la ville A pour se rendre à la ville B et revenir ensuite à la ville A. La première voiture roule à 40 km/h pendant tout le trajet. La seconde roule à 60 km/h à l'aller et à une autre vitesse constante au retour. Si les deux voitures reviennent au point de départ en même temps, quelle fut la vitesse de la seconde voiture au retour ?

2. Le périmètre d'un hexagone régulier H est identique à celui d'un triangle équilatéral T. Trouver le rapport de l'aire de H à celle de T.

3. Certains entiers s'expriment comme la somme d'entiers positifs impairs consécutifs. Par exemple, $64 = 13+15+17+19$. Le nombre 2009 s'exprime-t-il sous cette forme ? Si oui, trouver toutes les expressions possibles de 2009 sous cette forme.

4. Dans le diagramme ci-dessous on trouve trois carrées et trois angles x, y, et z. Trouver la somme des angles x, y, et z.



5. Soient x_1, x_2, x_3, x_4, et x_5 des nombres réels satisfaisant aux équations suivantes.

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 4x_2 & + & 9x_3 & + & 16x_4 & + & 25x_5 & = & 1\,, \\ 4x_1 & + & 9x_2 & + & 16x_3 & + & 25x_4 & + & 36x_5 & = & 8\,, \\ 9x_1 & + & 16x_2 & + & 25x_3 & + & 36x_4 & + & 49x_5 & = & 23\,. \end{array}$$

Évaluer $x_1 + x_2 + x_3 + x_4 + x_5$.

6. Un professeur écrit au tableau l'équation $x^2 - Ax + B = 0$ où A et B sont des entiers positifs et B est un nombre de deux chiffres. Une étudiante, en copiant l'équation, transpose les deux chiffres de B et transpose également les signes plus et moins. Malgré ces erreurs, elle trouve que son équation possède une racine, r, en commun avec l'équation correcte. Trouver toutes les valeurs possibles de A, B, et r.

Next we give solutions to the questions of the Math Kangaroo Contest Practice Set given at [2009 : 1-6].

1. (Grades 3-4) In the addition example, each letter represents a digit. Equal digits are represented by the same letter. Different digits are represented by different letters. Which digit does the letter K represent?

	O	K
$+$	K	O
W	O	W

(A) 0 (B) 1 (C) 2 (D) 8 (E) 9

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

The sum must be less than 200 since $99 + 99 = 198$. Thus, $W = 1$. Since the sum in the unit column is different from the sum in the tens column, the former must yield a carry. Thus $K+O = 11$. Trial and error now quickly reveals that $O = 2$ and $K = 9$.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; ELISA KUAN, student, Meadowridge School, Maple Ridge, BC; IAN CHEN, student, Centennial Secondary School, Coquitlam, BC; and LENA CHOI, student, École Banting Middle School, Coquitlam, BC.

Trial and error can be avoided: Once $K + O = 11$ is known, the sum in the tens column is then $1 + O + K = 12$, so $O = 2$ and, thus, $K = 9$.

2. (Grades 5-6) Ten caterpillars, arranged in a row one behind another, walked in the park. The length of each caterpillar was equal to 8 cm, and the distance any two adjacent caterpillars kept for safety reasons was 2 cm. What is the total length of their row?

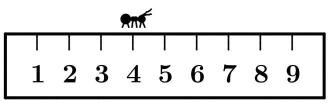
(A) 100 cm (B) 98 cm (C) 82 cm (D) 102 cm (E) 96 cm

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Ten caterpillars and nine spaces yields $(10 \times 8) + (9 \times 2) = 98$ cm.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; ELISA KUAN, student, Meadowridge School, Maple Ridge, BC; IAN CHEN, student, Centennial Secondary School, Coquitlam, BC; KEVIN LI, student, Pinetree Secondary School, Coquitlam, BC; and LENA CHOI, student, École Banting Middle School, Coquitlam, BC.

3. (Grades 7-8) An ant is running along a ruler of length 10 cm with a constant speed of 1 cm per second (see the figure). Any time when the ant reaches one of the ends of the ruler, it turns back and runs in the opposite direction. It takes the ant exactly 1 second to make a turn. The ant starts from the left end of the ruler. Nearest which number will it be after 2009 seconds?



(A) 1 cm (B) 2 cm (C) 3 cm (D) 4 cm (E) 5 cm

Solution by Cindy Chen, student, Burnaby North Secondary School, Burnaby, BC.

Walking the length of the ruler and turning around takes the ant 11 seconds. Now $2009 = 182 \cdot 11 + 7$. After $182 \cdot 11$ seconds, the ant will again be at the left end of the ruler since 182 is even. Thus, after 2009 seconds, the ant will be at the 7 cm mark. The closest given number is then 5 cm.

Also solved by IAN CHEN, student, Centennial Secondary School, Coquitlam, BC.

4. (Grades 9-10) Which of the numbers 2^6, 3^5, 4^4, 5^3, 6^2 is the greatest?

(A) 2^6 (B) 3^5 (C) 4^4 (D) 5^3 (E) 6^2

Solution by Kevin Li, student, Pinetree Secondary School, Coquitlam, BC.

Since $2^6 = 64$, $3^5 = 243$, $4^4 = 256$, $5^3 = 125$, and $6^2 = 36$, the largest clearly is 4^4.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; ELISA KUAN, student, Meadowridge School, Maple Ridge, BC; and LENA CHOI, student, École Banting Middle School, Coquitlam, BC.

5. (Grades 11-12) A decorator has prepared a mixed paint, in which the volumes of red and yellow colours were in the ratio $2 : 3$. The resulting colour seemed too light to him, so he added 2 L of red paint. This way, the ratio of the volumes of the red and yellow colours changed to $3 : 2$. How many litres of paint did the decorator use?

(A) 5 L (B) 6 L (C) 7 L (D) 8 L (E) 9 L

Solution by the editors.

Say the original amounts of paint were $2x$ and $3x$. Then the amount of red in the final product is $2x + 2$ while the amount of yellow is $3x$. Thus $\frac{2x+2}{3x} = \frac{3}{2}$. Solving the equation yields $x = \frac{4}{5}$, so the total amount of paint is $2x + 2 + 3x = 5x + 2 = 6$.

6. (Grades 3-4) Two boys are playing tennis until one of them wins four times. A tennis match cannot end in a draw. What is the greatest number of games they can play?

(A) 8 (B) 7 (C) 6 (D) 5 (E) 9

Solution by Alison Tam, student, Burnaby South Secondary School, Burnaby, BC.

They can play at most seven games; for example $ABABABA$.

Also solved by CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; KEVIN LI, student, Pinetree Secondary School, Coquitlam, BC; LENA CHOI, student, École Banting Middle School, Coquitlam, BC.

7. (Grades 5-6) In two years, my son will be twice as old as he was two years ago. In three years, my daughter will be three times as old as she was three years ago. Which of the following best describes the ages of the daughter and the son?

(A) The son is older; (B) The daughter is older; (C) They are twins;
(D) The son is twice as old as the daughter;
(E) The daughter is twice as old as the son.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

One easily finds that both children are six years old and therefore twins.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; and LENA CHOI, student, École Banting Middle School, Coquitlam, BC.

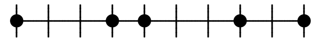
To find the children's ages without resorting to guess-and-check, say the son is now x years old. Then two years ago he was $x - 2$ years old, and in two years he will be $x + 2$ years old. Thus $2(x - 2) = x + 2$, which yields that $x = 6$. Likewise for the daughter.

8. (Grades 7-8) Some points are marked on a straight line so that all distances 1 cm, 2 cm, 3 cm, 4 cm, 5 cm, 6 cm, 7 cm, and 9 cm are among the distances between these points. At least how many points are marked on the line?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Solution by the editors.

Only $\binom{4}{2} = {}_4C_2 = \frac{4!}{2!\,2!} = 6$ distances are possible with four points. You need eight different distances, so at least five points are needed. The diagram shows that five points are sufficient.



9. (Grades 9-10) Eva, Betty, Linda, and Cathy went to the cinema. Since it was not possible to buy four seats next to each other, they bought tickets for seats number 7 and 8 in the 10^{th} row and tickets for seats number 3 and 4 in the 12^{th} row. How many seating arrangements can they choose from, if Cathy does not want to sit next to Betty?

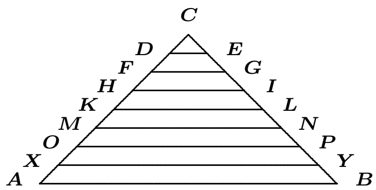
(A) 24 (B) 20 (C) 16 (D) 12 (E) 8

Solution Lena Choi, student, École Banting Middle School, Coquitlam, BC.

Cathy can sit with Linda and Eva with Betty in eight ways: $CLEB$, $CLBE$, $LCEB$, $LCBE$, $EBCL$, $EBLC$, $BECL$, and $BELC$. Likewise, Cathy can sit with Betty and Eva with Linda in eight ways. That makes 16 arrangements in all.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; and CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC.

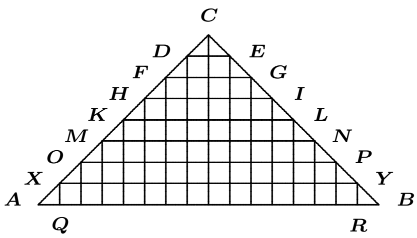
10. (Grades 11-12) Triangle ABC is isosceles with $BC = AC$. The segments DE, FG, HI, KL, MN, OP, and XY divide the sides AC and CB into equal parts. Find XY, if $AB = 40$ cm.



(A) 38 cm (B) 35 cm
(C) 33 cm (D) 30 cm (E) 27 cm

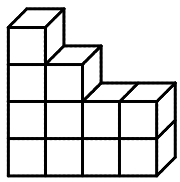
Solution by Ian Chen, student, Centennial Secondary School, Coquitlam, BC.

From C and each of the points on AC and BC drop a line orthogonally onto AB. Since the points on AC and BC are equally spaced, the intersection points on AB are also equally spaced. Thus $|AQ| = |BR| = \frac{1}{16}|AB| = \frac{5}{2}$. Therefore, $|XY| = |QR| = 40 - \frac{5}{2} \cdot 2 = 35$.

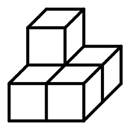


Also solved by CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; ELISA KUAN, student, Meadowridge School, Maple Ridge, BC; and LENA CHOI, student, École Banting Middle School, Coquitlam, BC.

11. (Grades 3-4) Matt and Nick constructed two buildings, shown in the figures, using identical cubes. Matt's building weighs 200 g, and Nick's building weighs 600 g. How many cubes from Nick's building are hidden and cannot be seen in the figure?



Nick's building



Matt's building

(A) 1 (B) 2 (C) 3
(D) 4 (E) 5

Solution by Elisa Kuan, student, Meadowridge School, Maple Ridge, BC.

Matt's building is built from exactly five cubes, so each cube weighs $\frac{1}{5}(200\,\text{g}) = 40\,\text{g}$. Therefore, Nick's building must consist of $600\,\text{g}/40\,\text{g} = 15$ cubes. Of these eleven are visible, so four are hidden.

Also solved by ALISON TAM, student, Burnaby South Secondary School, Burnaby, BC; CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; IAN CHEN, student, Centennial Secondary School, Coquitlam, BC; KEVIN LI, student, Pinetree Secondary School, Coquitlam, BC; and LENA CHOI, student, École Banting Middle School, Coquitlam, BC.

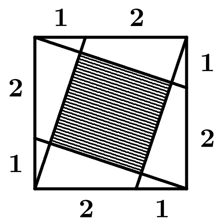
12. (Grades 5-6) Consider all four-digit numbers divisible by 6 whose digits are in increasing order, from left to right. What is the hundreds digit of the largest such number?

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

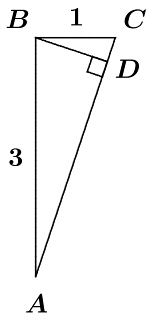
For the number to be divisible by 6, its unit digit must be even so at most 8. Therefore the thousands digit is at most 5. If the thousands digit is 5, the number must be 5678 which is not divisible by 3 and so not by 6. If the thousands digit is 4, the unit digit must still be 8 for the number to be even. Only the numbers 4568, 4578, and 4678 are like this. Of these only 4578 is divisible by 3 (and so by 6), so 4578 is the desired number.

13. (Grades 7-8) A square of side length 3 is divided by several segments into polygons as shown in the figure. What percent of the area of the original square is the area of the shaded figure?



(A) 30% (B) $33\frac{1}{3}\%$ (C) 35%
(D) 40% (E) 50%

Solution by the editors.



Consider the portion of the diagram on the left. By the Pythagorean Theorem, we have $|AC| = \sqrt{10}$. Since $\triangle ABC$ and $\triangle ADB$ are similar, $\frac{|BD|}{3} = \frac{1}{\sqrt{10}}$, so $|BD| = \frac{3}{\sqrt{10}}$. Using the Pythagorean Theorem again, $|AD| = \sqrt{3^2 - \left(\frac{3}{\sqrt{10}}\right)^2} = \sqrt{\frac{81}{10}} = \frac{9}{\sqrt{10}}$. It follows that $\triangle ABD$ has area $\frac{1}{2}\,|AD|\,|BD| = \frac{27}{20}$.

Note that the white region in the original diagram is exactly four copies of $\triangle ABD$. Therefore, the area of the white region is $\frac{27}{5}$ and the area of the shaded square is $3^2 - \frac{27}{5} = \frac{18}{5}$, which is 40% of the large square.

14. (Grades 9-10) A boy always tells the truth on Thursdays and Fridays, always tells lies on Tuesdays, and tells either truth or lies on the rest of the days of the week. Every day he was asked what his name was and six times in a row he gave the following answers: John, Bob, John, Bob, Pit, Bob. What did he answer on the seventh day?

(A) John (B) Bob (C) Pit (D) Kate
(E) Not enough information to decide

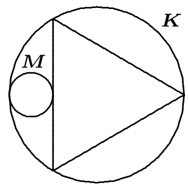
Solution by Ian Chen, student, Centennial Secondary School, Coquitlam, BC.

Since the boy tells the truth on Thursdays and Fridays, he must give the same answer twice in a row. So far the boy has never answered the same twice in a row, so his seventh answer must either be "Bob" to match the sixth answer or "John" to match the first answer. If his seventh answer is "Bob", then the seventh day is a Friday, so the fourth day was a Tuesday on which he lied and said "Bob." This is a contradiction. Thus the seventh answer is "John". The table below demonstrates that this is indeed possible.

John	Bob	John	Bob	Pit	Bob	**John**
Friday	Saturday	Sunday	Monday	Tuesday	Wednesday	Thursday
the truth	either	either	either	a lie	either	the truth

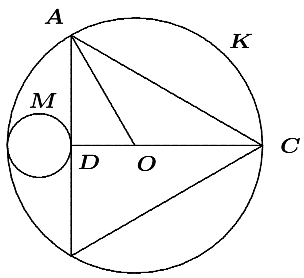
Also solved by CINDY CHEN, student, Burnaby North Secondary School, Burnaby, BC.

15. (Grades 11-12) An equilateral triangle and a circle M are inscribed in a circle K, as shown in the figure. What is the ratio of the area of K to the area of M?



(A) $8 : 1$ (B) $10 : 1$ (C) $12 : 1$
(D) $14 : 1$ (E) $16 : 1$

Solution by Ian Chen, student, Centennial Secondary School, Coquitlam, BC.



Let O be the centre of the triangle and suppose that the circle K has radius 1 and, thus, area π. Since the triangle is equilateral, $\angle DAO = 30°$, so $|OD| = |AO| \sin 30° = \frac{1}{2}$ and hence the diameter of circle M is $\frac{1}{2}$. Therefore, the area of the circle M is $\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{16}$ and the ratio of areas is $16 : 1$.

This issue's prize for the best solutions goes to Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany. We hope that all our readers enjoy the challenge of solving our problems and presenting their solutions to others.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

*Please send your solutions to the problems in this edition by **1 February 2010**. Solutions received after this date will only be considered if there is time before publication of the solutions. The Mayhem Staff ask that each solution be submitted on a separate page and that the solver's name and contact information be included with each solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

M407. *Proposed by Neven Jurič, Zagreb, Croatia.*

Determine whether or not the square at right can be completed to form a 4×4 magic square using the integers from 1 to 16. (In a magic square, the sums of the numbers in each row, in each column, and in each of the two main diagonals are all equal.)

			12
	16	1	10
	2	15	8

M408. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Determine all three-digit positive integers $\underline{abc}$ that satisfy the equation $\underline{abc} = \underline{ab} + \underline{bc} + \underline{ca}$. (Here $\underline{abc}$ denotes the three-digit positive integer with hundreds digit a, tens digit b, and units digit c.)

M409. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL and the Mayhem Staff.*

The three altitudes of a triangle lie along the lines $y = x$, $y = -2x + 3$, and $x = 1$. If one of the vertices of the triangle is at $(5, 5)$, determine the coordinates of the other two vertices.

M410. *Proposed by Matthew Babbitt, student, Albany Area Math Circle, Fort Edward, NY, USA.*

A cube with edge length a, a regular tetrahedron with edge length b, and a regular octahedron with edge length c all have the same surface area. Determine the value of $\frac{\sqrt{bc}}{a}$.

M411. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Triangle ABC has side lengths a, b, and c. If

$$2a + 3b + 4c \;=\; 4\sqrt{2a-2} \;+\; 6\sqrt{3b-3} \;+\; 8\sqrt{4c-4} \;-\; 20\,,$$

prove that triangle ABC is right-angled.

M412. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

For a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Determine all real numbers x for which $\lfloor x \rfloor \cdot \{x\} = x$.

....

M407. *Proposé par Neven Jurič, Zagreb, Croatie.*

Décider si oui ou non le carré ci-contre peut être complété en un carré magique de 4×4 en utilisant les entiers de 1 à 16. (Dans un carré magique, les sommes des nombres dans chaque ligne, chaque colonne et chaque diagonale principale sont toutes égales.)

			12
	16	1	10
	2	15	8

M408. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Trouver tous les entiers positifs de trois chiffres $\underline{abc}$ satisfaisant l'équation $\underline{abc} = \underline{ab} + \underline{bc} + \underline{ca}$. (On désigne ici par $\underline{abc}$ l'entier de trois chiffres dont le chiffre des centaines est a, celui des dizaines b et celui des unités c.)

M409. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL et l'Equipe de Mayhem.*

Les trois hauteurs d'un triangle sont sur les droites d'équation $y = x$, $y = -2x + 3$, et $x = 1$. Si l'un des sommets du triangle a comme coordonnées $(5, 5)$, trouver celles des deux autres sommets.

M410. *Proposé par Matthew Babbitt, étudiant, Albany Area Math Circle, Fort Edward, NY, É-U.*

Un cube d'arête a, un tétrahèdre régulier d'arête b et un octahèdre régulier d'arête c ont tous la même surface. Trouver la valeur de $\frac{\sqrt{bc}}{a}$.

M411. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Soit a, b et c les côtés du triangle ABC. Si

$$2a + 3b + 4c \;=\; 4\sqrt{2a-2} \;+\; 6\sqrt{3b-3} \;+\; 8\sqrt{4c-4} \;-\; 20\,,$$

montrer que ABC est un triangle rectangle.

M412. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Si x est un nombre réel, on désigne par $\lfloor x \rfloor$ le plus grand entier plus petit ou égal à x, et par $\{x\} = x - \lfloor x \rfloor$ la partie fractionnaire de x. Trouver tous les nombres réels x pour lesquels $\lfloor x \rfloor \cdot \{x\} = x$.

Mayhem Solutions

M376. *Proposed by the Mayhem Staff.*

Determine the value of x if $\left(10^{2009}+25\right)^2 - \left(10^{2009}-25\right)^2 = 10^x$.

Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.

Expanding, we have

$$\begin{aligned}
&\left(10^{2009}+25\right)^2 - \left(10^{2009}-25\right)^2 \\
&\quad= \left[\left(10^{2009}\right)^2 + 2\cdot 25\cdot 10^{2009} + 25^2\right] \\
&\qquad - \left[\left(10^{2009}\right)^2 - 2\cdot 25\cdot 10^{2009} + 25^2\right] \\
&\quad= 4\cdot 25\cdot 10^{2009} \\
&\quad= 100\cdot 10^{2009} \\
&\quad= 10^{2011}\,,
\end{aligned}$$

and so $x = 2011$.

Alternatively, factoring the left side as a difference of squares, we have

$$\begin{aligned}&\left(10^{2009}+25\right)^2-\left(10^{2009}-25\right)^2\\ &= \left[\left(10^{2009}+25\right)-\left(10^{2009}-25\right)\right]\cdot\left[\left(10^{2009}+25\right)+\left(10^{2009}-25\right)\right]\\ &= 50\cdot\left(2\cdot 10^{2009}\right)\\ &= 100\cdot 10^{2009} = 10^{2011},\end{aligned}$$

and so $x = 2011$, as above.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; LYLE BERSTEIN, student, Roslyn High School, Roslyn, NY, USA; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; KATHERINE JANELL EYRE and EMILY HENDRYX, students, Angelo State University, San Angelo, TX, USA; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.

M377. *Proposed by the Mayhem Staff.*

An arithmetic sequence consists of 9 positive integers. The sum of the terms in the sequence is greater than 200 and less than 220. If the second term in the sequence is 12, determine the sequence.

Solution by Katherine Janell Eyre and Emily Hendryx, students, Angelo State University, San Angelo, TX, USA.

Suppose that a is the first term in the sequence and d is the difference between successive terms. Then using formulae for arithmetic sequences and series, the n^{th} term can be written as $a_n = a + (n-1)d$ and the sum of the first n terms can be written as $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$. Hence,

$$S_9 = \frac{9}{2}(2a+8d) = 9(a+4d).$$

Also, we know that $a_2 = 12 = a + d$, from which we obtain $d = 12 - a$. Combining these, we get

$$S_9 = 9(a+4d) = 9\left(a+4(12-a)\right) = 9(48-3a) = 27(16-a).$$

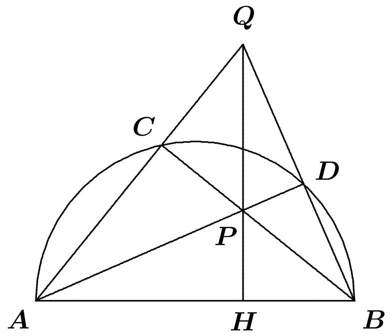
Since $200 < S_9 < 220$, we get $200 < 27(16-a) < 220$ which reduces to $7\frac{11}{27} < 16 - a < 8\frac{4}{27}$, or $7\frac{23}{27} < a < 8\frac{16}{27}$. Since a has to be a positive integer, we have $a = 8$, and so $d = 12 - a = 4$. We can then check that the resulting sequence satisfies all the requirements. Therefore, the sequence is 8, 12, 16, 20, 24, 28, 32, 36, 40.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MATTHEW BABBITT, *student, Albany Area Math Circle, Fort Edward, NY, USA; LYLE BERSTEIN, student, Roslyn High School, Roslyn, NY, USA; JACLYN CHANG, student, Western Canada High School, Calgary, AB; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There was one incomplete solution submitted.*

M378. *Proposed by the Mayhem Staff.*

Points C and D are chosen on the semicircle with diameter AB so that C is closer to A. Segments CB and DA intersect at P; segments AC and BD extended intersect at Q. Prove that QP extended is perpendicular to AB.

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.



Since AB is a diameter, then $\angle ACB = \angle ADB = 90°$. Thus, $\angle QCP = \angle QDP = 90°$, so quadrilateral $CPDQ$ is cyclic since a pair of opposite angles adds to $180°$. Note that $ACDB$ is also cyclic.

Let H be the intersection of PQ extended and AB. Then we have $\angle ABC = \angle ADC$, since each is subtended by the same arc. As well, $\angle PDC = \angle CQP$ since $CPDQ$ is a cyclic quadrilateral. Therefore, $\angle ADC = \angle PDC = \angle CQP = \angle AQH$. Hence, $\angle ABC = \angle AQH$.

But $\angle CAB = \angle QAH$ is a common angle in $\triangle ABC$ and $\triangle QAB$, and so $\triangle AQH$ is similar to $\triangle ABC$. Therefore, $\angle AHQ = \angle ACB = 90°$, and so QP extended is perpendicular to AB.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; LYLE BERSTEIN, student, Roslyn High School, Roslyn, NY, USA; ANTONIO GODOY TOHARIA, Madrid, Spain; R. LAUMEN, Deurne, Belgium; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania.

Most solvers noted that AD and BC are altitudes of $\triangle ABQ$. Thus, their point of intersection, P, is the orthocentre of the triangle and so the line segment through Q and P must be the third altitude of the triangle, and so is perpendicular to AB, as required.

M379. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The integers $27+C$, $555+C$, and $1371+C$ are all perfect squares, the square roots of which form an arithmetic sequence. Determine all possible values of C.

Solution by Matthew Babbitt, student, Albany Area Math Circle, Fort Edward, NY, USA.

Since the three given expressions have square roots in arithmetic sequence, then we may suppose that the square roots of $27+C$, $555+C$, and $1371+C$ are $a-d$, a, and $a+d$, respectively, for some a and d. Thus, $(a-d)^2 = 27+C$, $a^2 = 555+C$, and $(a+d)^2 = 1371+C$.

Note that

$$\begin{aligned}(a+d)^2+(a-d)^2-2a^2 &= \left(a^2+2ad+d^2\right)+\left(a^2-2ad+d^2\right)-2a^2 \\ &= 2d^2\,,\end{aligned}$$

and so

$$(1371+C) \;+\; (27+C) \;-\; 2(555+C) \;=\; 2d^2\,,$$

or $288 = 2d^2$, or $d^2 = 144$, which yields $d = \pm 12$.

If $d = 12$, then $(a+d)^2 = 1371+C$ and $(a+d)^2 = a^2+2ad+d^2 = 555+C+24a+144$. Combining these equations yields $24a = 672$, or $a = 28$. Thus, $C = a^2-555 = 28^2-555 = 229$.

If $d = -12$, then $(a+d)^2 = 1371+C$ and $(a+d)^2 = a^2-2ad+d^2 = 555+C-24a+144$. Combining these equations yields $-24a = 672$, or $a = -28$. Thus, $C = a^2-555 = (-28)^2-555 = 229$.

In either case, $C = 229$. We can check by substitution that $C = 229$ has the desired properties.

Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There was one incorrect solution submitted.

M380. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Triangle ABC is right-angled at C and has $BC = a$ and $CA = b$, with $a \geq b$. Squares $ABDE$, $BCFG$, and $CAHI$ are drawn externally to triangle ABC. The lines through FI and EH intersect at P, the lines through FI and DG intersect at Q, and the lines through DG and EH intersect at R. If triangle PQR is right-angled, determine the value of $\frac{b}{a}$.

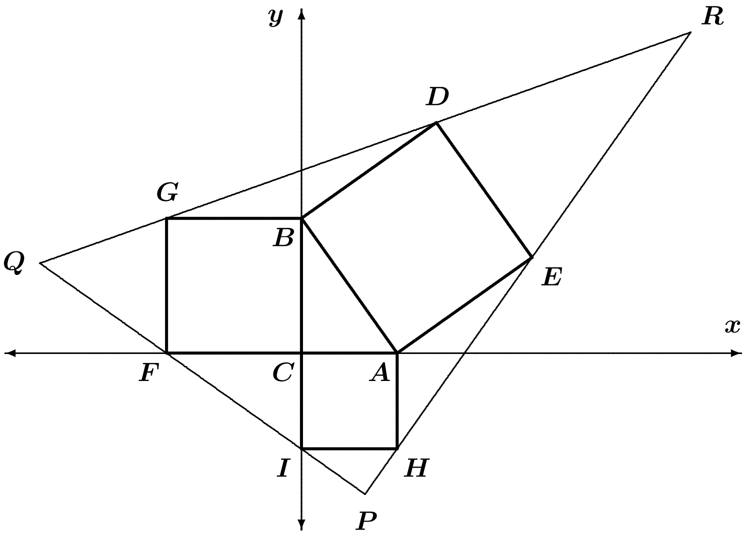
Composite of solutions by George Apostolopoulos, Messolonghi, Greece and Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON, modified by the editor.

We coordinatize the diagram on the following page by placing C at $(0,0)$, B at $(0,a)$ and A at $(b,0)$. Note that $a \geq b > 0$.

Since $BCFG$ is a square external to $\triangle ABC$, then it has side length a and has sides parallel to the axes, so the coordinates of G are $(-a, a)$ and the coordinates of F are $(-a, 0)$.

Similarly, since $CAHI$ is a square external to $\triangle ABC$, the coordinates of H are $(b, -b)$ and the coordinates of I are $(0, -b)$.

Since $ABDE$ is a square external to $\triangle ABC$ and the vector $\overrightarrow{AB}$ equals $(b, -a)$, then the vectors $\overrightarrow{BD}$ and $\overrightarrow{AE}$ are both equal to (a, b) (because they are perpendicular to AB and equal in length). Thus, the coordinates of D are $(a, a + b)$ and the coordinates of E are $(a + b, b)$.



The slope of QP equals the slope of FI. The coordinates of F are $(-a, 0)$ and the coordinates of I are $(0, -b)$, so the slope of QP is $-\frac{b}{a}$.

The slope of QR equals the slope of GD. The coordinates of G are $(-a, a)$ and the coordinates of D are $(a, a + b)$, so the slope of QR is $\frac{b}{2a}$.

The slope of PR equals the slope of HE. The coordinates of H are $(b, -b)$ and the coordinates of E are $(a + b, b)$, so the slope of PR is $\frac{2b}{a}$.

For two line segments to be perpendicular, the product of their slopes must equal -1. Since the slopes of QR and PR are both positive, then these segments are not perpendicular and the angle at R is not a right angle.

If there was a right angle at Q, then QR and QP would be perpendicular, or $-\frac{b}{a} \cdot \frac{b}{2a} = -1$, or $b^2 = 2a^2$, contradicting the fact that $a \geq b > 0$.

Thus, if $\triangle PQR$ has a right angle, then it can only occur at P. In that case, the segments QP and PR are perpendicular, so $-\frac{b}{a} \cdot \frac{2b}{a} = -1$, or $\frac{b^2}{a^2} = \frac{1}{2}$. Since $a \geq b > 0$, then $\frac{b}{a} = \frac{1}{\sqrt{2}}$.

Also solved by RICARD PEIRÓ, IES "Abastos", Valencia, Spain.

Problem of the Month

Ian VanderBurgh

This month, we'll look at another problem that is accessible to talented younger students, but that can still give pause to those with more experience.

Problem (2009 Gauss Contest, Grade 8) A list of six positive integers p, q, r, s, t, u satisfies $p < q < r < s < t < u$. There are exactly 15 pairs of numbers that can be formed by choosing two different numbers from this list. The sums of these 15 pairs of numbers are:

$$25,\ 30,\ 38,\ 41,\ 49,\ 52,\ 54,\ 63,\ 68,\ 76,\ 79,\ 90,\ 95,\ 103,\ 117\,.$$

Which sum equals $r + s$?

(A) 52 (B) 54 (C) 63 (D) 68 (E) 76

Now, we can do this the hard way, or we can do this the easy way. Of course, the hard way is actually easier than the easy way (which is pretty hard). Got that? Great – let's begin with the easy way.

Solution 1 The Big Idea: Because p, q, r, s, t, and u, are given to us in order of size, we should actually be able to deduce which sum corresponds with which pair in at least a few cases; if we're lucky, we'll be able match enough sums to pairs to determine which sum equals $r + s$. (Of course, since r and s are the "middle" numbers, then we'd expect their sum to be pretty much in the middle, so 63 is probably a good guess; since this is a multiple choice problem, 63 is almost certain to be the wrong answer.)

Let's get started. Can we tell which of the pairs should give the smallest sum? Yes! Since p and q are the smallest two numbers in the list, their sum must be the smallest of all of the sums. In other words, $p + q = 25$.

Can we tell which of the pairs should give the largest sum? Yes! Since t and u are the largest two numbers in the list, then their sum should be the largest of all of the sums. In other words, $t + u = 117$.

Next, we'll use the facts that $q = 25 - p$ and $t = 117 - u$ to do a little housekeeping and reduce the number of unknowns in our list to four. We thus rewrite our list as p, $25 - p$, r, s, $117 - u$, u.

Are there other pairs that we can track down easily? We had some success starting at the ends of the list of pairs, so let's keep trying to work inwards from the ends of the list of sums. Can we tell which of the pairs should give the second smallest sum? It's not $p + q$, since we've already used this one. We might guess $q + r$ or $p + r$. In fact, $p + r$ is smaller than

$q+r$, which we can actually deduce by using a chart:

$$\begin{array}{lllll} p+q & & & & \\ p+r & q+r & & & \\ p+s & q+s & r+s & & \\ p+t & q+t & r+t & s+t & \\ p+u & q+u & r+u & s+u & t+u \end{array}$$

In the chart, when we move down a fixed column, the sums increase since the first summand stays the same and the second increases; when we move to the right along a fixed row, the sums increase because the first summand increases and the second stays the same. Therefore, $p+r$ is smaller than $q+r$ (since it's immediately to the left) and is in fact smaller than any other sum except $p+q$. Therefore, $p+r$ is the second smallest sum, so $p+r=30$.

Similarly, the second largest sum must be $s+u$, so $s+u=103$.

We can now do some more housekeeping, writing $r = 30-p$ and $s=103-u$ to make our list p, $25-p$, $30-p$, $103-u$, $117-u$, u.

Now we come to a fork in the road. The third smallest total, according to our snazzy chart, is either $p+s$ or $q+r$. (Can you see why?) So either $p+s=p+103-u=38$ (that is, $p=u-65$) or $q+r=25-p+30-p=38$. In the second case, $55-2p=38$ or $2p=17$. But p is an integer, so this is not the case. Therefore, $p=u-65$, and our list becomes

$$p=u-65\,,\; q=90-u\,,\; r=95-u\,,\; s=103-u\,,\; t=117-u\,,\; u\,.$$

We have written each unknown in terms of one variable, which is undoubtedly a good thing.

Do you feel like we're zeroing in on the answer? (I'm not so sure myself!) Let's consolidate. We can at this point calculate some of the pairs, since we've got each unknown in terms of u. We can match up any unknown whose expression includes a "u" with any unknown whose expression includes a "$-u$" to actually get a numerical sum:

$$\begin{array}{ll} p+q=(u-65)+(90-u)=25\,; & q+u=(90-u)+u=90\,; \\ p+r=(u-65)+(95-u)=30\,; & r+u=(95-u)+u=95\,; \\ p+s=(u-65)+(103-u)=38\,; & s+u=(103-u)+u=103\,; \\ p+t=(u-65)+(117-u)=52\,; & t+u=(117-u)+u=117\,. \end{array}$$

Let's redraw our chart, but this time expressing the remaining pairs in terms of u only:

$$\begin{array}{lllll} 25 & & & & \\ 30 & 185-2u & & & \\ 38 & 193-2u & 198-2u & & \\ 52 & 207-2u & 212-2u & 220-2u & \\ 2u-65 & 90 & 95 & 103 & 117 \end{array}$$

Phew! The remaining numerical sums that are not yet visible in the chart are 41, 49, 54, 63, 68, 76, and 79.

Using our "over and down" idea from above, we see that the smallest remaining numerical sum, 41, cannot occur in the first column, because it is smaller than 52.

Therefore, 41 occurs at the top of the second column, and this yields $q + r = 185 - 2u = 41$, or $2u = 144$, or $u = 72$. It then follows that $r + s = 198 - 2u = 198 - 144 = 54$. The correct answer is (B). ■

At this point, you may feel like taking a holiday after this "easy way" before proceeding to the hard way! A couple of quick notes to cleanse the palate.

First, since $u = 72$, we could determine the values of the six original unknown integers to be 7, 18, 23, 31, 45, and 72. (We could then double check that the list of sums of pairs is correct.)

Second, we could have skipped the last two paragraphs of Solution 1 by making the following observation:

> Writing $r + s$ in terms of u, we obtain $r + s = 198 - 2u$. Is this even or odd? Since u is a positive integer, $r + s$ is even. Which of the remaining "unknown" sums are even? We can actually tell by looking at their representations in terms of u. The even ones are $r + s = 198 - 2u$, $r + t = 212 - 2u$, and $s + t = 220 - 2u$. Of these, $r + s$ is the smallest. (Why?) Thus, $r + s$ must equal the smallest remaining even sum, so $r + s = 54$.

Enough of the easy way! On to the hard way, but please bear with me – the hard way is actually pretty easy.

Solution 2 We easily deduced in Solution 1 that $p + q = 25$ and $t + u = 117$. We also made a chart that listed all of the expressions for the sums of each of the pairs. We also have a list of all of the numerical values the sums (expressions) can have. Let's add up both of these lists.

When we add up the 15 expressions in the chart in Solution 1, we get $5p + 5q + 5r + 5s + 5t + 5u$. First question: is it surprising that each unknown appears the same number of times? Second question: could you have figured out this sum without having to write out all of the pairs?

When we add up the known numerical values for the 15 sums, we get 980. Therefore, $5(p+q+r+s+t+u) = 980$ and $p+q+r+s+t+u = 196$. Thus, $r + s = 196 - (p + q) - (t + u) = 196 - 25 - 117 = 54$, as before. ■

A couple of thoughts about all of this to wrap up. First, we notice in the second solution that we never had to figure out the values of any of the unknowns. (We could avoid this in the first solution too by using the addendum.) Second, there's the issue of "easy" and "hard". The first solution, to me, requires more persistence than insight. We basically ground away until things fell apart. We didn't have to do anything hard, but it took a lot of effort. The second solution, to my mind, does require a couple of nice insights. Coming up with these insights is not easy, but once we have the right idea, the solution is actually pretty straightforward. Of course, this is what problem solving is all about!

THE OLYMPIAD CORNER

No. 281

R.E. Woodrow

We begin with the 24th Iranian Mathematical Olympiad 2006–2007 and the First Round problems. Thanks go to Bill Sands, Canadian Team Leader to the 48th IMO in Vietnam in 2007, for collecting them for our use.

24th IRANIAN MATHEMATICAL OLYMPIAD
First Round

1. Given integers $m > 2$ and $n > 2$, prove there is a sequence of integers a_0, a_1, $\dots$, a_k such that $a_0 = m$, $a_k = n$, and $(a_i + a_{i+1}) \mid (a_i a_{i+1} + 1)$ for each $i = 0, 1, \dots, k-1$.

2. Let I_1, I_2, $\dots$, I_n be n closed intervals of $\mathbb{R}$ such that among any k of them there are two with nonempty intersection. Prove that one can choose $k-1$ points in $\mathbb{R}$ such that each of the intervals contains at least one of the chosen points.

3. Let A, B, C, and D be four points on a circle ω and occurring on the circumference in that order. Prove that there exist four points M_1, M_2, M_3, and M_4 on ω which form a quadrilateral with perpendicular diagonals and are such that for each $i \in \{1, 2, 3, 4\}$

$$\frac{AM_i}{BM_i} = \frac{DM_i}{CM_i}\,.$$

4. Find all two-variable polynomials $p(x, y)$ with real coefficients such that $p(x+y, x-y) = 2p(x, y)$ for all real numbers x and y.

5. Let ω_1 and ω_2 be two circles such that the centre of ω_1 is located on ω_2. If the circles intersect at M and N, AB is an arbitrary diameter of ω_1, and A_1 and B_1 are the second intersections of AM and BN with the circle ω_2 (respectively), prove that A_1B_1 is equal to the radius of ω_1.

6. A stack of n books, numbered 1, 2, $\dots$, n is given. In a single round we make n moves, where the the i^{th} move in a round consists of turning over the i books at the top, taking them as a single block in the course of turning them over. After each round we start a new round similar to the previous one. Show that the initial arrangement will appear again after some number of rounds.

Continuing with the 24th Iranian Mathematical Olympiad, we give the Second Round. Thanks again go to Bill Sands, Canadian Team Leader to the 48th IMO in Vietnam in 2007, for collecting them.

24th IRANIAN MATHEMATICAL OLYMPIAD
Second Round

1. A regular polyhedron is a polyhedron which is convex and all of its faces are regular polygons. We call a regular polyhedron a *TLP* if and only if none of its faces is a triangle.

(a) Prove that each TLP can be inscribed in a sphere.

(b) Prove that the faces of each TLP are polygons of at most 3 different kinds. (That is, there are m, n, and p such that each face of the TLP is a regular n-gon, m-gon, or p-gon.)

(c) Prove that there is exactly one TLP with only pentagonal and hexagonal faces (the soccer ball).

(d) For $n > 3$, a prism which has 2 regular n-gons and n squares as its faces, is a TLP. Prove that except for these TLP's, there are only finitely many other TLP's.

2. A fluid flows in an infinite, line-shaped pipe. If a molecule of the fluid is at the point with coordinate x, then after t seconds it will be at the point with coordinate $P(t,x)$. Prove that if $P(t,x)$ is a polynomial in t and x, then all molecules are moving with a unique and constant speed.

3. Let C be a convex subset of $\mathbb{R}^3$ with positive volume. Suppose that C_1, C_2, ..., C_n are n translated (but not rotated) copies of C such that $C_i \cap C \neq \emptyset$ for each i, and such that C_i and C_j intersect at most on the boundary whenever $i \neq j$.

(a) Prove that if C is symmetric, then $n \leq 27$ and this is best possible.

(b) Prove the same thing for an arbitrary convex subset C.

4. A finite number of disjoint shapes in the plane are given. A *convex partition* of the plane is a collection of convex parts (subsets) such that the parts can intersect at most on their boundaries, the parts cover the plane, and each part contains exactly one of the shapes.

For which of the following sets of shapes does a convex partition exist?

(a) A finite number of distinct points.

(b) A finite number of disjoint line segments.

(c) A finite number of disjoint circular disks.

5. For $A \subseteq \mathbb{Z}$ and a, $b \in \mathbb{Z}$, let $aA + b = \{ax + b \colon x \in A\}$. If $a \neq 0$, then we say that $aA + b$ is similar to A. The Cantor Set, C, is the set of all nonnegative integers which have no digit 1 in their base 3 representation.

A representation of C is a partition of C into a finite number of two or more sets similar to C. That is, $C = \cup_{i=1}^{n} C_i$, where $n > 1$, the set $C_i = a_i C + b_i$ is similar to C for each i , and $C_i \cap C_j = \emptyset$ whenever $i \neq j$.

Note that $C = (3C) \cup (3C + 2)$ is a representation of C, and so is $C = (3C) \cup (9C + 2) \cup (9C + 6)$. A representation of C is a *primitive representation* if and only if the union of fewer than n of the C_i's is not similar to nor equal to C.

Prove that

(a) In a primitive representation of C, we have $a_i > 1$ for each i.

(b) In a primitive representation of C, each a_i is a power of 3.

(c) In a primitive representation of C, we have $a_i > b_i$ for each i.

(d) The only primitive representation of C is $C = (3C) \cup (3C + 2)$.

6. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients and let $P(x)$ be monic (that is, $P(x)$ has leading coefficient 1). Prove that there exists a monic polynomial $R(x)$ with integer coefficients such that $P(x) \mid Q\big(R(x)\big)$.

To complete the 24th Iranian Mathematical Olympiad, we now give the Third Round, with thanks to Bill Sands, Canadian Team Leader to the 48th IMO in Vietnam in 2007, for collecting them.

24th IRANIAN MATHEMATICAL OLYMPIAD
Third Round

1. Let A be a largest subset of $\{1, 2, \ldots, n\}$ such that each element of A divides at most one other element of A. Prove that

$$\frac{2n}{3} \leq |A| \leq 3\left\lceil \frac{n}{4} \right\rceil .$$

2. Does there exist a sequence of positive integers a_0, a_1, a_2, $\ldots$ such that $\gcd(a_i, a_j) = 1$ whenever $i \neq j$, and for all n the polynomial $\sum\limits_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$?

3. Triangle ABC is isosceles with $AB = AC$. The line ℓ passes through A and is parallel to BC. The points P and Q are on the perpendicular bisectors of AB and AC, respectively, and such that $PQ \perp BC$. The points M and N are on ℓ and such that $\angle APM$ and $\angle AQN$ are right angles. Prove that

$$\frac{1}{AM} + \frac{1}{AN} \leq \frac{2}{AB} .$$

4. Suppose that n lines are placed in the plane, such that no two are parallel and no three are concurrent. For each two lines let the angle between them be the smallest angle produced at their intersection. Find the largest value of the sum of the $\binom{n}{2}$ angles between the lines.

5. The point O is inside triangle ABC and such that $OA = OB + OC$. Let B' and C' be the midpoints of the arcs AOC and AOB, respectively. Prove that the circumcircles of COC' and BOB' are tangent to each other.

6. Find all polynomials $p(x)$ of degree 3 such that for all nonnegative real numbers x and y

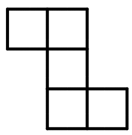
$$p(x+y) \ \geq \ p(x) + p(y)\,.$$

Next we give the two days of the 11th Mathematical Olympiad of Bosnia and Herzegovina, May 2006. Thanks go to Šefket Arslanagić, University of Sarajevo, Bosnia and Herzegovina, for sending them for our use.

11th MATHEMATICAL OLYMPIAD OF BOSNIA AND HERZEGOVINA
May, 2006
First Day

1. A Z-tile is any tile congruent to the one shown at right.

What is the least number of Z-tiles needed to cover an 8×8 grid, if every square of a Z-tile coincides with a square of the grid or is outside the grid. (The Z-tiles can overlap.)



2. Triangle ABC is given. Determine the set of the centres of all rectangles inscribed in the triangle ABC so that one side of the rectangle lies on the side AB of the triangle ABC.

3. Prove that for every positive integer n the inequality $\left\{n\sqrt{7}\right\} > \dfrac{3\sqrt{7}}{14n}$ holds, where $\{x\}$ is the fractional part of the real number x. (If $\lfloor x \rfloor$ is the greatest integer not exceeding x, then $\{x\} = x - \lfloor x \rfloor$.)

Second Day

4. For any two positive integers a and d prove that the infinite arithmetic progression

$$a,\ a+d,\ a+2d,\ \dots,\ a+nd,\ \dots$$

contains an infinite geometric progression of the form

$$b,\ bq,\ bq^2,\ \dots,\ bq^n,\ \dots\,,$$

where b and q are also positive integers.

5. The acute triangle ABC is inscribed in a circle with centre O. Let P be a point on the arc $\widehat{AB}$, where $C \notin \widehat{AB}$. The perpendicular from the point P to the line BO cuts the side AB at point S and the side BC at point T. The perpedicular from the point P to the line AO cuts the side AB at point Q and the side AC at point R. Prove that:

(a) The triangle PQS is isosceles.

(b) $PQ^2 = QR \cdot ST$.

6. Let $a_1, a_2, \ldots, a_n$ be real constants and for each real number x let

$$f(x) = \cos(a_1 + x) + \frac{\cos(a_2 + x)}{2} + \frac{\cos(a_3 + x)}{2^2} + \cdots + \frac{\cos(a_n + x)}{2^{n-1}}.$$

If $f(x_1) = f(x_2) = 0$, prove that $x_1 - x_2 = m\pi$, where m is an integer.

We finish this number with problems of the Vietnamese Mathematical Olympiad 2006–2007. Thanks go to Bill Sands, Canadian Team Leader to the 48th IMO in Vietnam in 2007, for collecting them.

VIETNAMESE MATHEMATICAL OLYMPIAD 2006–2007

February 8, 2007

Time: 3 hours

1. Solve the system of equations

$$\begin{aligned} 1 - \frac{12}{y+3x} &= \frac{2}{\sqrt{x}}, \\ 1 + \frac{12}{y+3x} &= \frac{6}{\sqrt{y}}. \end{aligned}$$

2. Let x and y be integers with $x \neq -1$ and $y \neq -1$, and such that

$$\frac{x^4 - 1}{y + 1} + \frac{y^4 - 1}{x + 1}$$

is also an integer. Prove that $x^4 y^{44} - 1$ is divisible by $x + 1$.

3. Triangle ABC has two fixed vertices, B and C, while the third vertex A is allowed to vary. Let H and G be the orthocentre and the centroid of ABC, respectively. Find the locus of A such that the midpoint K of the segment HG lies on the line BC.

4. Given a regular 2007-gon, find the smallest positive integer k satisfying the following property: In every set of k vertices there are 4 vertices which form a quadrilateral with three edges of the given 2007-gon.

5. Let b be a positive real number. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x+y) \; = \; f(x) \cdot 3^{b^y + f(y) - 1} \; + \; b^x \left(3^{b^y + f(y) - 1} - b^y \right)$$

for all real numbers x and y.

6. A trapezoid $ABCD$ with $BC \| AC$ and $BC > AD$ is inscribed in a circle k with centre O. The point P varies on the line BC outside the segment BC such that PA is not tangent to k. The circle with diameter PD intersects k at $E \neq D$. The lines BC and DE meet at M, and PA intersects k at $N \neq A$. Prove that the line MN passes through a fixed point.

7. Let $a > 2$ be a real number and

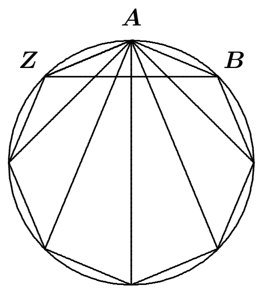
$$f_n(x) \; = \; a^{10} x^{n+10} + x^n + x^{n-1} + \cdots + x + 1$$

for each positive integer n. Prove that for each n the equation $f_n(x) = a$ has exactly one real root $x_n \in (0, \infty)$, and that the sequence $\{x_n\}_{n=1}^{\infty}$ has a finite limit as n approaches infinity.

We begin the solutions section of this number of the *Corner* with a continuation of solutions from our readers to problems of the Olimpiada Matemática Española 2005 given at [2008 : 341–342].

2. A triangle is said to be *multiplicative* if the product of the lengths of two of its sides equals the length of the third side.

Let $AB \ldots Z$ be a regular polygon with n sides, each of length 1. The $n - 3$ diagonals from the vertex A divide the triangle ZAB into $n - 2$ smaller triangles. Prove that all of these triangles are multiplicative.



Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the version of Zvonaru.

Let $\alpha = \frac{\pi}{n}$, $T_0 = Z$, and $T_{n-2} = B$. As we traverse the segment ZB from Z to B, let the intersection points of the $n-3$ diagonals with the segment ZB be T_1, T_2, $\ldots$, T_{n-3}. In $\triangle AZT_k$ we then have $\angle AZT_k = \alpha$, $\angle ZAT_k = k\alpha$, and $\angle ZT_kA = \pi - (k+1)\alpha$.

By the Law of Sines in $\triangle AZT_k$, we obtain

$$\frac{ZT_k}{\sin k\alpha} = \frac{AT_k}{\sin\alpha} = \frac{1}{\sin(k+1)\alpha},$$

hence, $AT_k = \dfrac{\sin\alpha}{\sin(k+1)\alpha}$ and $ZT_k = \dfrac{\sin k\alpha}{\sin(k+1)\alpha}$ for each k. We will prove that $\triangle AT_{k-1}T_k$ is multiplicative by proving that $AT_{k-1}\cdot AT_k = T_{k-1}T_k$.

Since $\sin^2 a = \dfrac{1-\cos 2a}{2}$ and $\sin a\sin b = \frac{1}{2}\big(\cos(a-b)-\cos(a+b)\big)$, we obtain

$$\begin{aligned} & AT_{k-1}\cdot AT_k = T_{k-1}T_k \\ \iff\ & AT_{k-1}AT_k = ZT_k - ZT_{k-1} \\ \iff\ & \frac{\sin\alpha}{\sin k\alpha}\cdot\frac{\sin\alpha}{\sin(k+1)\alpha} = \frac{\sin k\alpha}{\sin(k+1)\alpha} - \frac{\sin(k+1)\alpha}{\sin k\alpha} \\ \iff\ & \sin^2\alpha = \sin^2 k\alpha - \sin(k-1)\alpha\cdot\sin(k+1)\alpha \\ \iff\ & 1-\cos 2\alpha = 1-\cos 2k\alpha - \cos 2\alpha + \cos 2k\alpha, \end{aligned}$$

and the last line is certainly true.

4. In triangle ABC the sides BC, AC, and AB have lengths a, b, and c, respectively, and a is the arithmetic mean of b and c. Let r and R be the radius of the incircle and circumcircle of ABC, respectively. Prove that:

(a) $0^\circ \le \angle BAC \le 60^\circ$.

(b) The altitude from A is three times the inradius r.

(c) The distance from the circumcentre of ABC to the side BC is $R-r$.

Solved by George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Titu Zvonaru, Comăneşti, Romania. We give Bataille's version.

(a) From the Law of Cosines and $2a = b+c$ we have

$$\cos(\angle BAC) = \frac{b^2+c^2-a^2}{2bc} = \frac{3b^2+3c^2-2bc}{8bc},$$

so it suffices to show that $\dfrac{3b^2+3c^2-2bc}{8bc} \ge \dfrac{1}{2}$.

This is readily rewritten as $3b^2+3c^2-6bc \ge 0$, or $3(b-c)^2 \ge 0$, which clearly holds. Thus, $0^\circ \le \angle BAC \le 60^\circ$.

(b) Let h be the altitude from A. Twice the area of $\triangle ABC$ is equal to both ah and $r(a+b+c) = r(3a)$. Hence, $h = 3r$.

(c) Let O be the circumcentre of $\triangle ABC$. Since $\angle BAC = A$ is acute, $\angle BOC = 2A$, and so the distance from O to BC is $R\cos A$. Thus, we wish to prove that $r = R(1-\cos A)$, that is, $r = 2R\sin^2\dfrac{A}{2}$. Now, we have

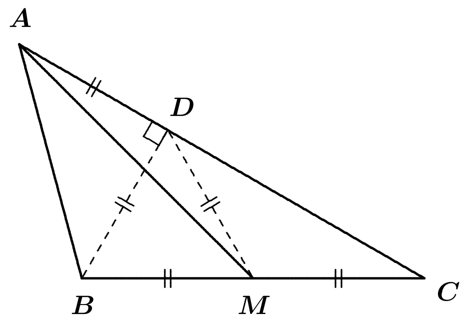
$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ (a well-known formula), and from $2a = b + c$ and the Law of Sines we obtain $2\sin A = \sin B + \sin C$, which easily transforms into $2\sin\frac{A}{2} = \cos\frac{B-C}{2}$. As a result

$$\begin{aligned} r &= 2R\sin\frac{A}{2}\left(2\sin\frac{B}{2}\sin\frac{C}{2}\right) \\ &= 2R\sin\frac{A}{2}\left(\cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right)\right) \\ &= 2R\sin\frac{A}{2}\left(\cos\left(\frac{B-C}{2}\right) - \sin\frac{A}{2}\right) \\ &= 2R\sin^2\frac{A}{2}, \end{aligned}$$

as desired.

5. In triangle ABC we have $\angle BAC = 45^\circ$ and $\angle ACB = 30^\circ$. Let M be the midpoint of the side BC. Prove that $\angle AMB = 45^\circ$ and that $BC \cdot AC = 2\,AM \cdot AB$.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comăneşti, Romania. We give Apostolopoulos' write-up.



If BD is perpendicular to AC, then $\angle ABD = 45^\circ$ and hence triangle ADB is isosceles and $AD = BD$. Note that the circumcircle of triangle BDC has M as its centre and BC as a diameter. So $\angle MDC = 30^\circ$ and $30^\circ = 2\angle MAD$, whence $\angle MAD = \angle AMD = 15^\circ$; $\angle AMB = 60^\circ - 15^\circ = 45^\circ$. By the Law of Sines

$$\frac{AB}{\sin 30^\circ} = \frac{BC}{\sin 45^\circ} \quad\text{and}\quad \frac{AC}{\sin 135^\circ} = \frac{AM}{\sin 30^\circ}.$$

Therefore,

$$\frac{AB}{BC} = \frac{1}{\sqrt{2}} \quad\text{and}\quad \frac{AM}{AC} = \frac{1}{\sqrt{2}},$$

so that

$$\frac{AB \cdot AM}{BC \cdot AC} = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2},$$

and $BC \cdot AC = 2AM \cdot AB$, as desired.

7. Prove that the equation $x^2 + y^2 - z^2 - x - 3y - z - 4 = 0$ has infinitely many integer solutions.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give Zvonaru's write-up.

The given equation is successively equivalent to

$$\begin{aligned} x^2 - x - 6 &= z^2 + z + \frac{1}{4} - y^2 + 3y - \frac{9}{4}\,; \\ (x-3)(x+2) &= \left(z + \frac{1}{2}\right)^2 - \left(y - \frac{3}{2}\right)^2\,; \\ (x-3)(x+2) &= (z+y-1)(z-y+2)\,. \end{aligned}$$

Taking $x = \alpha$ where α is an integer and equating

$$\begin{cases} z + y - 1 = \alpha - 3\,, \\ z - y + 2 = \alpha + 2\,, \end{cases}$$

we obtain infinitely many solutions $(x, y, z) = (\alpha, -1, \alpha - 1)$, where α is an arbitrary integer.

On the other hand, choosing

$$\begin{cases} z + y - 1 = \alpha + 2\,, \\ z - y + 2 = \alpha - 3\,, \end{cases}$$

we obtain the solutions $(x, y, z) = (\alpha, 4, \alpha - 1)$, with α an integer.

Now we return to readers' solutions to problems of the 54th Czech Mathematical Oympiad 2004/2005, Category B, 10th Class [2008: 342–344].

D1. Find all pairs (a, b) of real numbers such that each of the equations

$$\begin{aligned} x^2 + ax + b &= 0\,, \\ x^2 + (2a+1)x + 2b + 1 &= 0\,, \end{aligned}$$

has two distinct real roots and the roots of the second equation are reciprocals of the roots of the first equation.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution of Apostolopoulos, modified by the editor.

Note that $b \neq 0$, because the roots of the first equation are nonzero. If $x^2 + ax + b = 0$ has roots x_1 and x_2, then since each constant term of each quadratic is the product of the roots we have

$$\frac{1}{b} = \frac{1}{x_1 x_2} = \frac{1}{x_1} \cdot \frac{1}{x_2} = 2b + 1\,,$$

hence $2b^2 + b - 1 = 0$ and we deduce that $b = -1$ or $b = \frac{1}{2}$.

On the other hand, the roots of each quadratic sum to minus the coefficient of x, so that

$$-(2a+1) \;=\; \frac{1}{x_1}+\frac{1}{x_2} \;=\; \frac{x_1+x_2}{x_1x_2} \;=\; \frac{-a}{b}\,,$$

hence $a = b(2a+1)$.

Now, if $b = \frac{1}{2}$, then $a = b(2a+1)$ leads to the contradiction $0 = \frac{1}{2}$. However, taking $b = -1$ in the equation $a = b(2a+1)$ and then solving for a yields $(a,b) = \left(-\frac{1}{3}, -1\right)$, which one can check yields quadratics with the required properties.

D2. Let $ABCD$ be a parallelogram. A line through D meets the segment AC in G, the side BC in F, and the line AB in E. The triangles BEF and CGF have the same area. Determine the ratio $|AG| : |GC|$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We present the solution of Amengual Covas.

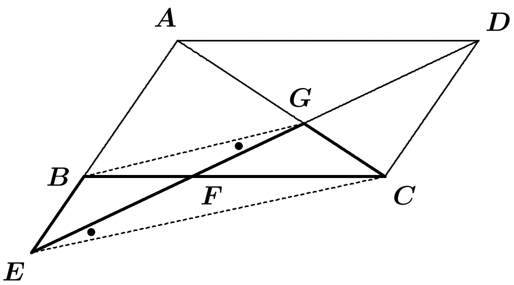
From similar triangles AGD, CGF and FBE, FCD we have

$$\begin{aligned}\frac{AG}{GC} \;&=\; \frac{AD}{FC} \;=\; \frac{BC}{FC} \;=\; \frac{BF+FC}{FC}\\ &=\; \frac{BF}{FC}+1 \;=\; \frac{BE}{CD}+1 \;=\; \frac{BE}{AB}+1\,.\end{aligned} \qquad (1)$$

Since the triangles BEF and CGF have the same area, it follows that $BF \cdot FE = GF \cdot FC$, that is, $\frac{BF}{GF} = \frac{FC}{FE}$.

Thus, in triangles BFG and CFE, the sides about equal angles BFG and CFE are proportional, implying the triangles are similar and the equal corresponding angles FGB and FEC imply that BG and EC are parallel. Hence,

$$\frac{AG}{GC} \;=\; \frac{AB}{BE}\,. \qquad (2)$$



From (1) and (2), it follows that $\frac{AG}{GC}$ satisfies the equation $\phi = \frac{1}{\phi}+1$. Therefore, the required ratio is the golden ratio:

$$\frac{AG}{GC} \;=\; \frac{1+\sqrt{5}}{2}\,.$$

D3. Let $k \geq 3$ be an integer. We have k piles of stones with (respectively) $1, 2, \ldots, k$ stones in them. At each turn we choose three piles, merge them together, and add one stone (not already in a pile) to the resulting pile. Prove that if after some number of turns only one pile remains, then the number of stones in that pile is not divisible by 3.

Solution by Titu Zvonaru, Comăneşti, Romania.

Denote the total number of stones at the start by $S = \frac{k(k+1)}{2}$. After the first turn we will have $k-2$ piles and $S+1$ stones. After the second turn we will have $k - 2 \cdot 2$ piles and $S+2$ stones. Continuing in this manner, after the p-th turn we will have $k - 2p$ piles and $S + p$ stones.

If after p turns only one pile remains, then we must have $k - 2p = 1$, that is, $k = 2p+1$ and the one pile will have

$$S + p \;=\; \frac{(2p+1)(2p+2)}{2} + p \;=\; (2p+1)(p+1) + p$$

stones in it. Thus, the one pile has $2p^2 + 4p + 1$ stones, and we have to prove that this number is not divisible by 3.

Write

$$2p^2 + 4p + 1 \;=\; 2(p-1)(p+1) + p + 3(p+1)\,.$$

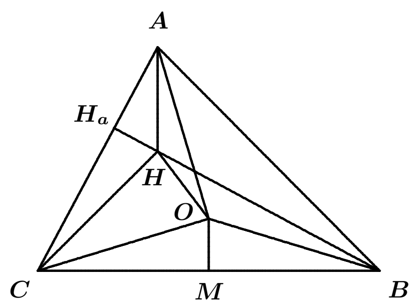
If p is divisible by 3, then $p-1$ and $p+1$ are not divisible by 3, hence $2(p-1)(p+1)$ is not divisible by 3 while $p + 3(p+1)$ is divisible by 3. Therefore, in this first case, $2p^2 + 4p + 1$ is not divisible by 3.

If p is not divisible by 3, then one of $p-1$ or $p+1$ is divisible by 3, hence $2(p-1)(p+1)$ is divisible by 3, hence $2(p-1)(p+1) + 3(p+1)$ is divisible by 3 and adding p to this last quantity gives a result not divisible by 3. Therefore, in this second case, $2p^2 + 4p + 1$ is not divisible by 3.

In all cases, the number of stones in the last pile is not divisible by 3.

D4. Let ABC be a scalene triangle with orthocentre H and circumcentre O. Prove that if $\angle ACB = 60^\circ$, then the bisector of $\angle ACB$ is the perpendicular bisector of OH.

Solved by Ricardo Barroso Campos, University of Seville, Seville, Spain; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Barroso Campos.



We have $\angle CBA = 60^\circ - \alpha$ and $\angle CAB = 60^\circ + \alpha$.

Furthermore, $\angle HAH_a = 30^\circ$ (since $AH \perp BC$), $\angle AHH_a = 60^\circ$, $\angle HCH_a = 30^\circ - \alpha$ (since $CH \perp AB$), and $\angle H_aHC = 60^\circ + \alpha$.

Also, $\angle COB = 2\angle CAB = 120^\circ + 2\alpha$ and $\angle AOB = 2\angle ACB = 120^\circ$. Since $\angle AOB = \angle AHB = 120^\circ$ the quadrilateral $AHOB$ is cyclic, and $\angle BHO = \angle BAO = 30^\circ$.

We now have

$$\begin{aligned} \angle CHO &= 360^\circ - \angle CHH_a - \angle H_aHA - \angle AHB - \angle BHO \\ &= 360^\circ - \left(60^\circ + \alpha\right) - 60^\circ - 120^\circ - 30^\circ \;=\; 90^\circ - \alpha \end{aligned}$$

$$\begin{aligned} \angle HCO &= \angle ACO - \angle ACH \;=\; \left(90^\circ - \angle ABC\right) - \left(30^\circ - \alpha\right) \\ &= 90^\circ - \left(60^\circ - \alpha\right) - 30^\circ + \alpha \;=\; 2\alpha\,, \end{aligned}$$

hence, $\angle COH = 180^\circ - \left(90^\circ - \alpha\right) - 2\alpha = 90^\circ - \alpha$.

Therefore, triangle HOC is isosceles with $CH = CO$, and the angle bisector at C is the perpendicular bisector of OH.

D5. Find all real numbers x such that

$$\frac{x}{x+4} \;=\; \frac{5\lfloor x\rfloor - 7}{7\lfloor x\rfloor - 5}\,,$$

where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x.

Solved by Titu Zvonaru, Comăneşti, Romania.

Writing $\alpha = \lfloor x\rfloor$ and solving for x, we obtain

$$x \;=\; \frac{10\alpha - 14}{\alpha + 1}\,.$$

Since $\alpha \le x < \alpha + 1$, we have

$$\begin{aligned} \alpha \;\le\; x \quad&\Longleftrightarrow\quad \frac{\alpha^2 + \alpha - 10\alpha + 14}{\alpha + 1} \;\le\; 0 \\ &\Longleftrightarrow\quad \frac{(\alpha-2)(\alpha-7)}{\alpha+1} \;\le\; 0\,; \end{aligned}$$

$$\begin{aligned} \alpha + 1 \;>\; x \quad&\Longleftrightarrow\quad \frac{\alpha^2 + 2\alpha + 1 - 10\alpha + 14}{\alpha + 1} \;>\; 0 \\ &\Longleftrightarrow\quad \frac{(\alpha-3)(\alpha-5)}{\alpha+1} \;>\; 0\,; \end{aligned}$$

From the first set of deductions, $\alpha \in (-\infty, -1) \cup [2, 7]$; from the second set of deductions, $\alpha \in (-1, 3) \cup (5, \infty)$; hence $\alpha \in [2, 3) \cup (5, 7]$. Since α is an integer, $\alpha = 2$ and $x = 2$; or $\alpha = 6$ and $x = \frac{46}{7}$; or $\alpha = 7$ and $x = 7$.

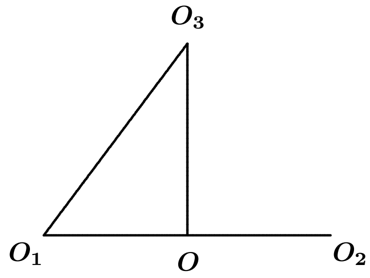
Therefore, all solutions are given by $x \in \left\{2, \frac{46}{7}, 7\right\}$.

D6. In a circle Γ with radius r are inscribed two mutually tangent circles, Γ_1 and Γ_2, each with radius $r/2$. Circle Γ_3 is tangent to Γ_1 and Γ_2 externally and to Γ internally. Circle Γ_4 is tangent to Γ_2 and Γ_3 externally and to Γ internally. Determine the radii of the circles Γ_3 and Γ_4.

Solution by Titu Zvonaru, Comăneşti, Romania.

(i) Let Γ have centre O and Γ_i have centre O_i for each i. Let Γ_3 have radius x.

We have $OO_1 = \frac{r}{2}$, $O_1O_3 = \frac{r}{2} + x$, and $OO_3 = r - x$. By the Pythagorean Theorem, $O_1O_3^2 = OO_1^2 + OO_3^2$, or

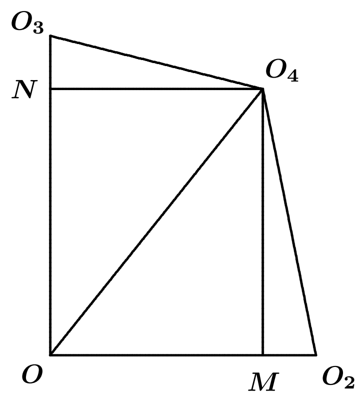


$$\frac{1}{4}r^2 + rx + x^2 = \frac{5}{4}r^2 + r^2 - 2rx + x^2 ,$$

hence, $3rx = r^2$ and Γ_3 has radius $\frac{1}{3}r$.

(ii) Denote by y the radius of Γ_4. We have

$$OO_2 = \frac{r}{2}; \qquad O_2O_4 = \frac{r}{2} + y;$$
$$O_4O_3 = \frac{r}{3} + y; \quad OO_3 = \frac{2r}{3};$$
$$OO_4 = r - y .$$



Let M be the projection of O_4 onto OO_2 and let N be the projection of O_4 onto OO_3. By the Pythagorean Theorem,

$$OO_4^2 - OM^2 = O_4O_2^2 - (OO_2 - OM)^2 ,$$

which upon substituting the above yields

$$r^2 + y^2 - 2ry - OM^2 = \frac{r^2}{4} + y^2 + ry - \frac{r^2}{4} + r \cdot OM - OM^2 ,$$

hence, $OM = r - 3y$.

Heron's formula $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$ for the area $[ABC]$ of $\triangle ABC$ (where $a = BC$, $b = AC$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$) yields

$$O_4N = \frac{2[OO_3O_4]}{OO_3} = \frac{2\sqrt{r \cdot \frac{r}{3}\left(\frac{2r}{3} - y\right) \cdot y}}{\left(\frac{2r}{3}\right)} = \sqrt{y(2r - 3y)} .$$

We now have $(r - 3y)^2 = OM^2 = O_4N^2 = y(2r - 3y)$, since ONO_4M is a rectangle. Hence, $(6y - r)(2y - r) = 0$, which yields $y \in \left\{\frac{r}{6}, \frac{r}{2}\right\}$.

However, $y < \frac{r}{2}$, therefore Γ_4 has radius $\frac{r}{6}$.

S2. Let ABC be a right triangle with $a = |BC|$, $b = |AC|$, and $a = |BC|$ and such that $a < b < c$. Let Q be the midpoint of BC and let S be the midpoint of AB. The line CA meets the perpendicular bisector of AB at R. Prove that $|RQ| = |RS|$ if and only if $a^2 : b^2 : c^2 = 1 : 2 : 3$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Ricardo Barroso Campos, University of Seville, Seville, Spain; and Titu Zvonaru, Comănești, Romania. We give the solution and comment of Amengual Covas.

Triangles ABC and ARS are similar, so we obtain

$$AR = \frac{AS}{AC}\cdot AB = \frac{c}{2b}AB = \frac{c^2}{2b}; \qquad (1)$$

$$CR = CA - AR = b - \frac{c^2}{2b} = \frac{2b^2-\big(a^2+b^2\big)}{2b} = \frac{b^2-a^2}{2b}, \qquad (2)$$

since $c^2 = a^2 + b^2$ holds by the Pythagorean Theorem in $\triangle ABC$.

Thus,

$$\begin{aligned}
& |RQ| = |RS| \iff RQ^2 = RS^2 \\
\iff\ & QC^2 + CR^2 = AR^2 - AS^2 \text{ (Pyth. Thm. in } \triangle QCR, \triangle ASR) \\
\iff\ & AR^2 - CR^2 = QC^2 + AS^2 \\
\iff\ & (AR+CR)(AR-CR) = QC^2 + AS^2 \\
\iff\ & AC(AR-CR) = QC^2 + AS^2 \\
\iff\ & AR - CR = \frac{QC^2+AS^2}{AC} = \frac{(\frac{a}{2})^2+(\frac{c}{2})^2}{b} = \frac{a^2+c^2}{4b} \\
\iff\ & AR = \frac{AR+CR}{2} + \frac{AR-CR}{2} = \frac{CA}{2} + \frac{AR-CR}{2} \\
& = \frac{b}{2} + \frac{a^2+c^2}{8b} = \frac{b}{2} + \frac{(c^2-b^2)+c^2}{8b} = \frac{3b^2+2c^2}{8b} \\
& \text{and} \\
& CR = \frac{AR+CR}{2} - \frac{AR-CR}{2} = \frac{CA}{2} - \frac{AR-CR}{2} \\
& = \frac{b}{2} - \frac{a^2+c^2}{8b} = \frac{b}{2} - \frac{a^2+(a^2+b^2)}{8b} = \frac{3b^2-2a^2}{8b} \\
\iff\ & \frac{c^2}{2b} = \frac{3b^2+2c^2}{8b} \quad \text{and} \quad \frac{b^2-a^2}{2b} = \frac{3b^2-2a^2}{8b} \\
& \text{(substituting } AR = \frac{c^2}{2b} \text{ and } CR = \frac{b^2-a^2}{2b} \text{ by (1) and (2), resp.)} \\
\iff\ & 3b^2 = 2c^2 \quad \text{and} \quad 2a^2 = b^2 \iff a^2 : b^2 : c^2 = 1 : 2 : 3\,.
\end{aligned}$$

Comment. For other properties of triangles with $a^2 : b^2 : c^2 = 1 : 2 : 3$, see Problem 3 of the 31st Spanish Mathematical Olympiad, First Round, given in [1998 : 452–453], with solution at [2000 : 143–144].

S3. Find all real numbers x such that

$$\left\lfloor \frac{x}{1-x} \right\rfloor = \frac{\lfloor x \rfloor}{1-\lfloor x \rfloor},$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

Solved by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give Wang's write-up.

Note first that

$$\frac{1}{1-\lfloor x \rfloor} - 1 = \frac{\lfloor x \rfloor}{1-\lfloor x \rfloor} = \left\lfloor \frac{x}{1-x} \right\rfloor = \left\lfloor \frac{1}{1-x} - 1 \right\rfloor = \left\lfloor \frac{1}{1-x} \right\rfloor - 1.$$

Hence, the given equation is equivalent to

$$\left\lfloor \frac{1}{1-x} \right\rfloor = \frac{1}{1-\lfloor x \rfloor}. \qquad (1)$$

Since the left side of (1) is an integer, we must have $1 - \lfloor x \rfloor = \pm 1$, that is, $\lfloor x \rfloor = 0$ or 2. If $\lfloor x \rfloor = 0$, then $x \in [0,1)$ while if $\lfloor x \rfloor = 2$, then $x \in [2,3)$.

However, if $x \in \left[\frac{1}{2}, 1\right)$, then $\frac{1}{1-\lfloor x \rfloor} = 1$, while $0 < 1-x \le \frac{1}{2}$ implies that $\frac{1}{1-x} \ge 2$, and so $\left\lfloor \frac{1}{1-x} \right\rfloor \ge 2$. Hence, in this case, (1) cannot hold.

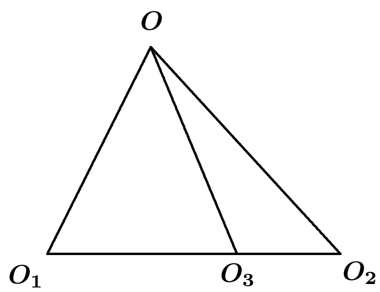
If $x \in \left[0, \frac{1}{2}\right)$, then $\lfloor x \rfloor = 0$. Also, $1 - x > \frac{1}{2}$ implies $1 \le \frac{1}{1-x} < 2$. Hence, (1) holds with a value of 1 on each side.

If $x \in [2,3)$, then $\lfloor x \rfloor = 2$. Also, $-2 < 1 - x \le -1$ implies that $-1 \le \frac{1}{1-x} < -\frac{1}{2}$. Hence, (1) holds with a value of -1 on each side.

By these three cases, the solution set is $\left[0, \frac{1}{2}\right) \cup [2,3)$.

K1. Circle Γ_1 with radius 1 is externally tangent to circle Γ_2 with radius 2. Each of the circles Γ_1 and Γ_2 is internally tangent to circle Γ_3 with radius 3. Determine the radius of the circle Γ, which is tangent externally to the circles Γ_1 and Γ_2 and internally to the circle Γ_3.

Solution by Titu Zvonaru, Comănești, Romania.



Let Γ have centre O and Γ_i have centre O_i for each i. Let Γ have radius x.

Note that

$OO_1 = 1 + x$; $OO_2 = 2 + x$; $O_2O_3 = 1$;
$OO_3 = 3 - x$; $O_1O_3 = 2$.

By Stewart's Theorem we obtain

$$OO_1^2 \cdot O_3O_2 - OO_3^2 \cdot O_1O_2 + OO_2^2 \cdot O_1O_3 = O_1O_3 \cdot O_3O_2 \cdot O_1O_2,$$

which upon substituting yields $1+2x+x^2-27-3x^2+18x+8+2x^2+8x = 6$. Upon solving for x, we obtain $x = \frac{24}{28} = \frac{6}{7}$.

K2. On a public website participants vote for the world's best hockey player of the last decade. The percentage of votes a player receives is rounded off to the nearest percent and displayed on the website. After Jožko votes for Miroslav Šatan, the hockey player's score of 7% remains unchanged. What is the minimum number of people (including Jožko) who voted? (Each participant votes exactly once and for a single player only.)

Solved by Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give the solution of Geupel.

Let N and m denote the total number of participants and the number of people who voted for Šatan, respectively. Jožko is counted in both N and m, so the given information yields

$$\begin{aligned} \frac{13}{200} \;&\le\; \frac{m-1}{N-1} < \frac{m}{N} \;\le\; \frac{15}{200}\,, \\ 13N+187 \;&\le\; 200m \;\le\; 15N\,, \end{aligned} \qquad (1)$$

thus, $187 \le 2N$, and hence $N \ge 94$. Writing $N = 94 + k$, where k is a nonnegative integer, and substituting this into (1), we obtain

$$1409 + 13k \;\le\; 200m \;\le\; 1410 + 15k\,.$$

It follows that $m \ge 8$ and $15k \ge 1600 - 1410 = 190$, so that $k \ge 13$.

We now check the case $k = 13$, $N = 94 + k = 107$, $m = 8$. By direct calculation of $\frac{m-1}{N-1} = \frac{7}{106} = 0.0666\ldots$ and $\frac{m}{N} = \frac{8}{107} = 0.074\ldots$, we readily see that $N = 107$ is indeed the minimum number of participants (where eight people voted for Šatan).

K4. Find all triples of real numbers x, y, z such that

$$\lfloor x\rfloor - y \;=\; 2\lfloor y\rfloor - z \;=\; 3\lfloor z\rfloor - x \;=\; \frac{2004}{2005}\,,$$

where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution by Apostolopoulos.

For convenience, let $k = \frac{2004}{2005}$. We then have

$$x = 3\lfloor z\rfloor - k\,; \quad y = \lfloor x\rfloor - k\,; \quad z = 2\lfloor y\rfloor - k\,. \qquad (1)$$

Since $0 < k < 1$, the preceding equations imply

$$\lfloor x\rfloor = 3\lfloor z\rfloor - 1\,; \quad \lfloor y\rfloor = \lfloor x\rfloor - 1\,; \quad \lfloor z\rfloor = 2\lfloor y\rfloor - 1\,.$$

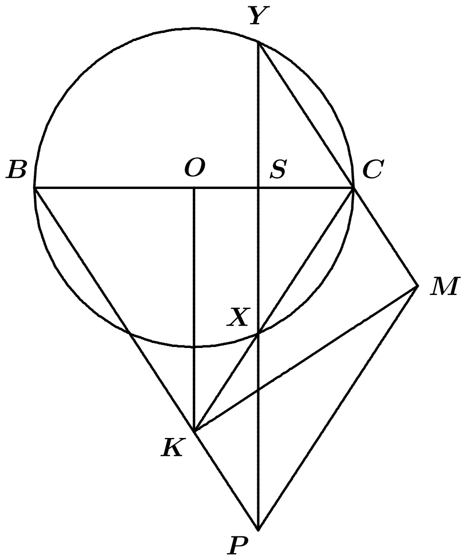
Thus, $\lfloor z\rfloor = 2\big(\lfloor x\rfloor - 1\big) - 1 = 2\lfloor x\rfloor - 3$ and $\lfloor x\rfloor = 3\big(2\lfloor x\rfloor - 3\big) - 1$, and solving for $\lfloor x\rfloor$ yields $\lfloor x\rfloor = 2$. Then $\lfloor y\rfloor = 2 - 1 = 1$ and $\lfloor z\rfloor = 2\cdot 1 - 1 = 1$.

Finally, substituting the values for $\lfloor x\rfloor$, $\lfloor y\rfloor$, and $\lfloor z\rfloor$ back into (1) gives the unique solution $(x,y,z) = \left(\frac{4011}{2005}, \frac{2006}{2005}, \frac{2006}{2005}\right)$.

Next we turn to readers' solutions to problems of the First Round, 23rd Iranian Mathematical Olympiad given in the *Corner* at [2008 : 344–345].

5. The segment BC is the diameter of a circle and XY is a chord perpendicular to BC. The points P and M are chosen on XY and CY, respectively, such that $CY \| PB$ and $CX \| MP$. Let K be the intersection of the lines CX and PB. Prove that $PB \perp MK$.

Solved by George Apostolopoulos, Messolonghi, Greece; and Titu Zvonaru, Comănești, Romania. We give the solution by Zvonaru.



Let O and S be the midpoints of BC and XY, respectively. It is easy to see that $CX = CY$.

Since $CY \| PB$, we have

$$\angle CBK = \angle BCY = \angle SCY = \angle SCX = \angle BCK .$$

It follows that $BK = CK$ and K lies on the perpendicular bisector of BC. Since $\triangle CXS$ and $\triangle CKO$ are similar,

$$CK = \frac{CX \cdot CO}{CS} . \qquad (1)$$

Also, $\triangle CSY$ and $\triangle PSB$ are similar, so it follows that

$$SP = \frac{BS \cdot YS}{CS} . \qquad (2)$$

By (2) we have $YP = SP + YS = \dfrac{YS(BS+CS)}{CS}$, hence $YP = 2 \cdot \dfrac{YS \cdot OC}{CS}$. Since $\triangle YCX$ is similar to $\triangle YMP$, we deduce that

$$YM = \frac{YC \cdot YP}{YX} \implies YM = \frac{YC}{2 \cdot YS} \cdot 2 \cdot \frac{YS \cdot OC}{CS} ,$$

hence,

$$YM = \frac{CY \cdot OC}{CS} . \qquad (3)$$

By (1) and (3), we have $BK = YM$, hence quadrilateral $BKMY$ is a parallelogram as $BK \| YM$. Since $\angle BYM = \angle BYC = 90^\circ$, the parallelogram $BKMY$ is a rectangle and $MK \perp PB$.

That completes this number of the *Corner*. Send me your nice solutions and generalizations.

BOOK REVIEWS

Amar Sodhi

Problems from Murray Klamkin: The Canadian Collection
Edited By Andy Liu and Bruce Shawyer,
Mathematical Association of America, 2009
ISBN 9780-88385-828-8, hardcover, 249+xvi pages, US$61.50
Reviewed by **Bill Sands**, *University of Calgary, Calgary, AB*

This book is a well-deserved tribute to the late great Murray Klamkin, long-time contributor to this journal (which in this review I will denote informally as ***Crux***), and its original Olympiad Corner Editor. The two Editors of this book, both mathematicians and problemists well-known to ***Crux*** readers and familiar with Murray's work, have assembled all of his problems that had been published in ***Crux*** over the years (which means 1977 to 2005, stretching over the first four ***Crux*** Editors), and their published solutions (usually by other readers), into one volume. The result is a must-have book for any fan of ***Crux***, or of math problems in general. Andy Liu, who was Murray's colleague at the University of Alberta for over twenty years, also contributes a memorial tribute that he originally wrote for ***Crux*** and that was first published in September 2005; it is reprinted at the beginning of this book. Bruce Shawyer is a former Editor of ***Crux*** who would have processed many of the contributions in this book when they first appeared in ***Crux***.

I'm a former Editor of ***Crux*** myself, and had the pleasure of reading Murray's ***Crux*** contributions for ten years, so browsing through this book was a bit of a walk down memory lane for me! In addition, Murray's problems were tackled by most of the ***Crux*** regulars of the time, and it was a pleasure to see the names of many old friends and read their thoughts again.

The 120 problems have been categorized into 15 sections depending on topic, and arranged chronologically in each section. The sections range in size from 3 to 19 problems, the latter being the section "Algebraic Inequalities", which will be no surprise to long-time ***Crux*** readers! There are also two additional special sections, one containing all 54 of Murray's "quickies" that appeared in ***Crux*** from time to time, the other containing the 15 problems published in the special September 2005 issue of ***Crux*** which were proposed by readers and dedicated to Murray. After these lists of problems come the solutions. At the end is an index of proposers and solvers (Murray of course excluded), and an index of the problems.

So of course my recommendation is that you all buy this book, even those of you who already own a complete run of ***Crux*** back to its birth as ***Eureka***.

To end this review here are some specific comments on (and corrections to) the content. I'll try to suppress the obsessive editor who still lurks inside me, and who cringes at a misplaced comma (and there are a few in this book). Instead I'll restrict myself to matters which may interest you as a reader of the book. I also won't list instances where the original ***Crux*** entry contained an error or omission which has been repeated in the present book, as the Editors cannot be faulted in such cases.

- Q40 (page 8): The inequalities are missing from this problem. They should be:

  (i) $\dfrac{ab}{c}+\dfrac{bc}{d}+\dfrac{cd}{a}+\dfrac{da}{b} \;\geq\; a+b+c+d$,

  (ii) $a^2b+b^2c+c^2d+d^2a \;\geq\; abc+bcd+cda+dab$.

- Problem 1734 (page 18): The first line of the problem is missing. It should be: "Determine the minimum value of".

- Problem 3024 (page 219): A large part of the proposers' solution, and all of two other solutions (as published in ***Crux***), has been left out.

- The original ***Crux*** references have not been given for the solutions to the Klamkin Tribute Problems. (Almost all of these solutions have appeared in the September 2006 issue.)

Crocheting Adventures with Hyperbolic Planes
By Daina Taimiņa, A.K. Peters, Ltd., 2009
ISBN 13: 978-1-56881-452-0, hardcover, 148+xi pages, US$35.95
Reviewed by ***J. Chris Fisher****, University of Regina, Regina, SK*

The appearance of "crocheting" and "hyperbolic planes" together in the title might seem somewhat incongruous, but these are, indeed, the featured topics in this delightful little book. Taimiņa quickly convinces us that her crocheted models promote a deeper understanding of hyperbolic geometry and of surfaces in general.

Riemann showed that the natural setting for each of the three classical plane geometries is a surface of constant curvature: the points of the plane are represented by the points of the surface, while the line joining two of those points becomes the curve between the corresponding points that traces the shortest distance along the surface. The Euclidean plane (with its unique parallel to a given line through a point not on that line) is flat—it has zero curvature. The elliptic plane (in which any two lines must intersect) can be represented by a sphere—or, more precisely, by a sphere whose antipodal points are identified; its curvature is the reciprocal of the square of the sphere's radius. But, there is a problem with the hyperbolic plane (in which more than one line through a point will be parallel to a given line not through that point): a hyperbolic plane is the geometry of a surface of constant negative curvature, but Hilbert proved that it is not possible in Euclidean 3-space to have an equation represent a surface extending indefinitely in all directions that has constant negative curvature. Hilbert's result is sometimes misinterpreted as implying that the hyperbolic plane is too big to fit in our Euclidean world; however, it does fit if you ruffle the surface as William Thurston demonstrated in the 1970's with a simple paper model. That model lacked the flexibility and durability required for classroom explorations, but it inspired the author to crochet a hyperbolic plane. In 1997 she used her models to help explain hyperbolic geometry in a class she taught at Cornell, where she is an adjunct associate professor. That class led to articles in mathematical and educational journals, to lectures, workshops and, eventually, to this book.

The author expects that crocheters can experience some of the excitement that comes from mathematics, while mathematicians can learn about a tool that helps to explore mathematical ideas. Consequently, her text avoids technical definitions and precise mathematical details. Taimiņa has a knack at providing intuitive descriptions for mathematical concepts. She explains, for example, that the curvature at a point of a surface can be determined by measuring the circumference of a small circle of radius r centred at that point (where, of course, the radius is measured on the surface): the circumference is greater than, equal to, or less than $2\pi r$, according as the curvature is negative, zero, or positive. As a consequence, if you flatten a disk that is taken from a surface of positive curvature, for example the peel of an orange, so that it lies in a plane, it splits apart and covers less area than the plane disk of the same radius; for negative curvature, loose-leafed lettuce for example, the surface covers more area than the Euclidean disk of the same radius, so that it overlaps itself when flattened. Among the many pictures in the text are examples of surfaces of negative curvature in nature; in fact, her models look a bit like blossoms or lettuce heads.

Apparently, for the purpose of model building, crocheting has an advantage over knitting by offering more control over the stitches. Disclaimer: This reviewer was forced by time constraints to go against the author's advice to crochet his own model. The book provides illustrated instructions to lead the novice from his first loop to the finished model, with careful explanations on how to ensure that the resulting hand-crafted object has constant negative curvature. It certainly sounds easy. Even without a model in hand, the most interesting chapter for me was "What Can You Learn from Your Model?" A straight line on any flexible surface of constant curvature is obtained by folding the surface without stretching it—think of folding a sheet of paper and observing the straight line that appears when the sheet is opened. The line can be marked by weaving a coloured strand of yarn along the fold. You can thus form a triangle on your hyperbolic model and observe that the angle sum is less than two right angles. Or you can mark a line and a point not on it, and observe that there are two lines through that point, one to the right and one to the left, that separate those lines through the point that intersect the given line from those lines through the point that miss the given line. It is clear that except for those boundary lines, a line that misses the given line diverges from it both to the right and to the left, while somewhere in between, where the two lines come closest to one another, there is a common perpendicular. Indeed, many of the basic results of hyperbolic geometry can be visualized in this way.

The book devotes three chapters to the history of geometry and of crocheting. These histories were too sketchy and idiosyncratic for my taste, but they do serve to provide a necessary context. The final chapter is somewhat more successful. It describes some of the many ways hyperbolic geometry has proved to be useful; beyond mathematics and computer science are applications to biology, chemistry, medicine, physics, and several other disciplines. One chapter investigates other surfaces of negative curvature. Two examples: a catenoid is the surface obtained by revolving a catenary (shaped like a chain hanging from its two fixed ends) about a horizontal axis below the chain; a helicoid is a spiral ramp whose boundary is a double helix. In fact, these two surfaces have the same intrinsic geometry (in the same way that the right circular cylinder and the plane have the same intrinsic geometry); the author demonstrates this remarkable theorem by cutting her model of the catenoid along one catenary and twisting it (without stretching) into a helicoid and back again. It is amazing how much information has been packed into this beautifully illustrated book of fewer than 150 pages.

Mathematical Mindbenders
By Peter Winkler, A.K. Peters, Ltd., 2007
ISBN 13: 978-1-56881-336-3, softcover, 148+x pages, US$24.95
Reviewed by **Amar Sodhi**, *Sir Wilfred Grenfell College, Corner Brook, NL*

Mathematical Mindbenders is a collection of puzzles which should appeal to those who enjoy problems of a recreational bent that require a certain degree of mathematical training. For example, one is asked to determine, without a direct calculation, a solitary missing digit in the nine-digit number 2^{29}. In a significantly more difficult problem, one is asked to determine the minimum area of a polygon having an odd number of sides of unit length. For good measure, Winkler also includes some open problems.

The book begins innocuously with a set of very gentle "Warm-ups" which gives little to suggest that the adjective Mathematical is needed in the title. All this changes however when Winkler resorts to "Stretching the Imagination". Among the eight challenges in this chapter is a dice problem (dating from 1978) due to George Sicherman and the much more recent, but quite well known, "Names in Boxes" puzzle. In the former, one is asked to construct two nonstandard dice which, as far as the sum is concerned, behaves as a pair of standard dice, while the latter requires a strategy to be developed giving the best chance that all 100 people find their own name when allowed to open up to 50 of 100 boxes containing these names.

Chapter titles such as: *Numerical Conundrums*, *Two Dimensions and Three*, *Lines and Graphs*, and *Games and Strategies* correctly suggest that this volume contains many problems that rely on a degree of knowledge in discrete mathematics and geometry, while the chapters *New Visits to Old Friends* and *Severe Challenges* offer a selection of familiar but somewhat difficult problems. Some succumb to a trial and error approach, but in others, clever counting, skillful use of polynomials and/or generating functions, logic, or inductive reasoning may be helpful. The proposed solutions for at least three of the problems use the Axiom of Choice.

There are two chapters of problems (mostly) devised by Winkler. The first one, *The Adventures of Ant Alice*, is a series of probability problems involving ants walking either on rods or circles. This chapter is also included in the recently released *A Lifetime of Puzzles: A Collection of Puzzles in Honor of Martin Gardner's 90th Birthday*. The other chapter, *A Wordy Digression: The Game of HIPE*, is rather intriguing in that it contains no mathematics whatsoever. Instead, Winkler introduces a word game that he, together with three other high-school juniors, invented while at a summer program hosted by the National Science Foundation in 1966.

In "HIPE" a player is given a string of consecutive letters (such as HIPE). The challenge is to find a word that contains the given string. It may be unusual to include a chapter for wordsmiths, but this just emphasizes the fact that this is a puzzle book aimed at the mathematician or aspiring mathematician rather than a mathematical puzzle book. Anybody having any doubts on this score need only go to the final chapter of the book where Winkler, rather informally, discusses (rather than poses) ten "Unsolved or Just Solved" problems. Among the "just solved" problems is John Conway's Angel Problem which appeared in Winkler's previous work, *Mathematical Puzzles: A Connoisseur's Collection*.

One aspect of the book which I did find irritating was the way in which Winkler intertwined "Sources" with "Solutions". Typically, after tackling a problem, I would go to "sources and solutions" at the end of the chapter to see what Winkler had to offer. Too many times I would have to wade through acknowledgements, written in a somewhat informal manner, in order to get to the solution on hand. Sometimes the clarity of the solution suffered because of this. There is every reason for Winkler to acknowledge those who have sent him problems and many of the sources provided allow the reader to discover even more puzzles, but this could have been achieved in a less obtrusive fashion. However, this criticism is somewhat churlish given the pleasure I derived from the book.

Winkler, overall, has selected a fine array of problems from numerous sources. There are a few puzzles of either a dubious nature or poorly posed, but with over ninety challenges in the book there is bound to be something to please the most demanding puzzle maven. My favourites are those which have featured in Martin Gardner's long discontinued *Mathematical Puzzles* column in *Scientific American*. Anyone enjoying *Mathematical Mindbenders* will surely find their own favourites.

Several Symmetric Inequalities of Exponential Kind

Arkady Alt

In this article we suggest a general approach for proving certain symmetric inequalities of exponential kind in three variables which have appeared in print at various times.

Theorem 1 Let n, m, p, and q be arbitrary nonnegative real numbers, such that $n \geq m$ and $p \geq q$. Then for any positive real numbers a, b, c the following inequality holds

$$\frac{a^{n+p}+b^{n+p}+c^{n+p}}{a^{m+q}+b^{m+q}+c^{m+q}} \geq \frac{a^n+b^n+c^n}{a^m+b^m+c^m} \cdot \frac{a^p+b^p+c^p}{a^q+b^q+c^q}\,.$$

Proof: Let $\sigma(x) = \sigma(x;a,b,c) = \sum\limits_{\text{cyclic}} a^x$; the inequality then becomes

$$\frac{\sigma\,(n+p)}{\sigma\,(m+q)} \geq \frac{\sigma\,(n)}{\sigma\,(m)} \cdot \frac{\sigma\,(p)}{\sigma\,(q)}\,.$$

The inequality is essentially the same upon switching n and p or m and q, so we may suppose that $n \geq p$ and $m \geq q$. Then $q = \min\{n, m, p, q\}$.

Since the inequality to be proved is equivalent to $\sigma\,(n+p)\,\sigma\,(m)\,\sigma\,(q) \geq \sigma\,(m+q)\,\sigma\,(n)\,\sigma\,(p)$ and we also have

$$\begin{aligned} &\sigma\,(n+p)\,\sigma\,(m)\,\sigma\,(q) \\ &= \sum_{\text{cyclic}} a^{n+p} \cdot \left(\sum_{\text{cyclic}} a^{m+q} + \sum_{\text{cyclic}} \left(a^m b^q + b^m a^q\right) \right) \\ &= \left(\sum_{\text{cyclic}} a^{n+p} \right) \left(\sum_{\text{cyclic}} a^{m+q} \right) + \sum_{\text{cyclic}} \left(a^{n+p}+b^{n+p}\right)\left(a^m b^q + b^m a^q\right) \\ &\quad + \sum_{\text{cyclic}} c^{n+p}\left(a^m b^q + b^m a^q\right), \end{aligned}$$

with the analogous equality holding for $\sigma\,(m+q)\,\sigma\,(n)\,\sigma\,(p)$, it therefore suffices to prove the following two inequalities:

$$\sum_{\text{cyclic}} \left(a^{n+p}+b^{n+p}\right)\left(a^m b^q + b^m a^q\right) \geq \sum_{\text{cyclic}} \left(a^{m+q}+b^{m+q}\right)\left(a^n b^p + b^n a^p\right),$$

$$\sum_{\text{cyclic}} c^{n+p}\left(a^m b^q + b^m a^q\right) \geq \sum_{\text{cyclic}} c^{m+q}\left(a^n b^p + b^n a^p\right).$$

Crux Mathematicorum with Mathematical Mayhem, Volume 35, Issue 7

The first inequality above is settled by the following calculation:

$$
\begin{aligned}
&\sum_{\text{cyclic}} \left(a^{n+p} + b^{n+p}\right)\left(a^m b^q + b^m a^q\right) \\
&\quad - \sum_{\text{cyclic}} \left(a^{m+q} + b^{m+q}\right)\left(a^n b^p + b^n a^p\right) \\
= &\sum_{\text{cyclic}} \Big(a^{n+p+m} b^q + b^{n+p+m} a^q + a^m b^{n+p+q} + b^m a^{n+p+q} \\
&\qquad - a^{n+m+q} b^p - b^{n+m+q} a^p - a^n b^{m+p+q} - b^n a^{m+p+q}\Big) \\
= &\sum_{\text{cyclic}} a^q b^q \left(a^{n+m+p-q} + b^{n+m+p-q} - a^{n+m} b^{p-q} - b^{n+m} a^{p-q}\right) \\
&\quad + \sum_{\text{cyclic}} a^m b^m \left(a^{n+p+q-m} + b^{n+p+q-m} - a^{p+q} b^{n-m} - b^{p+q} a^{n-m}\right) \\
= &\sum_{\text{cyclic}} a^q b^q \left(a^{n+m} - b^{n+m}\right)\left(a^{p-q} - b^{p-q}\right) \\
&\quad + \sum_{\text{cyclic}} a^m b^m \left(a^{p+q} - b^{p+q}\right)\left(a^{n-m} - b^{n-m}\right) \;\geq\; 0\,.
\end{aligned}
$$

Lastly, since

$$
\begin{aligned}
\sum_{\text{cyclic}} c^{n+p}\left(a^m b^q + b^m a^q\right) &= \sum_{\text{cyclic}} c^q \left(a^{n+p} b^m + b^{n+p} a^m\right); \\
\sum_{\text{cyclic}} c^{m+q}\left(a^n b^p + b^n a^p\right) &= \sum_{\text{cyclic}} c^q \left(a^{m+p} b^n + b^{m+p} a^n\right),
\end{aligned}
$$

the second inequality that remains to be proved now follows immediately from

$$
\begin{aligned}
&\sum_{\text{cyclic}} c^q \left(a^{n+p} b^m + b^{n+p} a^m - a^{m+p} b^n - b^{m+p} a^n\right) \\
= &\sum_{\text{cyclic}} a^m b^m c^q \left(a^{n-m+p} + b^{n-m+p} - a^p b^{n-m} - b^p a^{n-m}\right) \\
= &\sum_{\text{cyclic}} a^m b^m c^q \left(a^p - b^p\right)\left(a^{n-m} - b^{n-m}\right) \;\geq\; 0\,. \qquad \blacksquare
\end{aligned}
$$

Corollary 1 Let k be a nonnegative integer and let $p \geq q \geq 0$. Then for any positive real numbers a, b, and c the following inequality holds

$$
\frac{a^{kp} + b^{kp} + c^{kp}}{a^{kq} + b^{kq} + c^{kq}} \;\geq\; \left(\frac{a^p + b^p + c^p}{a^q + b^q + c^q}\right)^k .
$$

Proof: We set $n = kp$, $m = kq$ in Theorem 1 to obtain

$$
\frac{\sigma(kp+p)}{\sigma(kq+q)} \;\geq\; \frac{\sigma(kp)}{\sigma(kq)} \cdot \frac{\sigma(p)}{\sigma(q)}
$$

and that yields the inequality

$$\frac{\sigma\big((k+1)p\big)}{\sigma\big((k+1)q\big)}\left(\frac{\sigma(p)}{\sigma(q)}\right)^{-(k+1)} \;\geq\; \frac{\sigma(kp)}{\sigma(kq)}\left(\frac{\sigma(p)}{\sigma(q)}\right)^{-k},$$

which implies that

$$\frac{\sigma(kp)}{\sigma(kq)}\left(\frac{\sigma(p)}{\sigma(q)}\right)^{-k} \;\geq\; \frac{\sigma(1\cdot p)}{\sigma(1\cdot q)}\left(\frac{\sigma(p)}{\sigma(q)}\right)^{-1} \;=\; 1\,,$$

and the inequality to be proved now follows. ∎

Theorem 2 Let a, b, and c be positive real numbers. Then for any positive integer n the function

$$L_n(x) \;=\; L_n(x;a,b,c) \;=\; \frac{a^n+b^n+c^n}{a^{nx}+b^{nx}+c^{nx}}\sum_{\text{cyclic}}\left(\frac{a^x}{b+c}\right)^n$$

is increasing in x on $(0,\infty)$.

Proof: Let p, $q\in(0,\infty)$ and $q<p$. Due to the homogeneity of $L_n(x;a,b,c)$ with respect to a, b, and c, it suffices to prove the assertion when $a+b+c=1$. Using the expansion $\frac{1}{(1-t)^n}=\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}t^k$ we obtain

$$\begin{aligned}
&\frac{\sigma(np)\sigma(nq)}{\sigma(n)}\big(L_n(p)-L_n(q)\big)\\
&\quad=\;\sigma(nq)\sum_{\text{cyclic}}\frac{a^{np}}{(1-a)^n}\;-\;\sigma(np)\sum_{\text{cyclic}}\frac{a^{nq}}{(1-a)^n}\\
&\quad=\;\sigma(nq)\sum_{\text{cyclic}}\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}a^{k+np}-\sigma(np)\sum_{\text{cyclic}}\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}a^{k+nq}\\
&\quad=\;\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}\big(\sigma(nq)\sigma(k+np)-\sigma(np)\sigma(k+nq)\big)\\
&\quad=\;\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}\sum_{\text{cyclic}}\big(a^{k+np}b^{nq}+a^{nq}b^{k+np}-a^{k+nq}b^{np}-a^{np}b^{k+nq}\big)\\
&\quad=\;\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}\sum_{\text{cyclic}}a^{nq}b^{nq}\left(a^{n(p-q)}-b^{n(p-q)}\right)\left(a^k-b^k\right)\;\geq\;0\,,
\end{aligned}$$

since $\left(a^{n(p-q)}-b^{n(p-q)}\right)\left(a^k-b^k\right)\geq 0$ for any nonnegative integer k. ∎

Corollary 2 For any positive real numbers a, b, c, r and any positive numbers p and q such that $q<r<p$ the following inequality holds

$$\frac{1}{\sigma(nq)}\sum_{\text{cyclic}}\left(\frac{a^q}{b^r+c^r}\right)^n \leq \frac{1}{\sigma(nr)}\sum_{\text{cyclic}}\left(\frac{a^r}{b^r+c^r}\right)^n \leq \frac{1}{\sigma(np)}\sum_{\text{cyclic}}\left(\frac{a^p}{b^r+c^r}\right)^n.$$

Proof: Since $L_n\big(x; a^r, b^r, c^r\big)$ is increasing in x and $q < r < p$, we have

$$L_n\left(\frac{q}{r}; a^r, b^r, c^r\right) \;\leq\; L_n\left(1; a^r, b^r, c^r\right) \;\leq\; L_n\left(\frac{p}{r}; a^r, b^r, c^r\right),$$

which is equivalent to the inequality to be proved. ■

By the results of Corollary 1 and Corollary 2 we obtain successively

$$\frac{1}{\sigma(nq)}\sum_{\text{cyclic}}\left(\frac{a^q}{b^r + c^r}\right)^n \;\leq\; \frac{1}{\sigma(nr)}\sum_{\text{cyclic}}\left(\frac{a^r}{b^r + c^r}\right)^n ;$$

$$\frac{\displaystyle\sum_{\text{cyclic}}\left(\frac{a^q}{b^r + c^r}\right)^n}{\displaystyle\sum_{\text{cyclic}}\left(\frac{a^r}{b^r + c^r}\right)^n} \;\geq\; \frac{\sigma(nr)}{\sigma(nq)} \;\geq\; \left(\frac{\sigma(nr)}{\sigma(nq)}\right)^n ,$$

and similarly we obtain

$$\frac{\displaystyle\sum_{\text{cyclic}}\left(\frac{a^p}{b^r + c^r}\right)^n}{\displaystyle\sum_{\text{cyclic}}\left(\frac{a^r}{b^r + c^r}\right)^n} \;\geq\; \frac{\sigma(np)}{\sigma(nr)} \;\geq\; \left(\frac{\sigma(p)}{\sigma(r)}\right)^n .$$

It follows that for any positive real numbers a, b, c, r and any positive real numbers p, q such that $q < r < p$, the following inequality holds

$$\frac{1}{\sigma^n(q)}\sum_{\text{cyclic}}\left(\frac{a^q}{b^r + c^r}\right)^n \leq \frac{1}{\sigma^n(r)}\sum_{\text{cyclic}}\left(\frac{a^r}{b^r + c^r}\right)^n \leq \frac{1}{\sigma^n(p)}\sum_{\text{cyclic}}\left(\frac{a^p}{b^r + c^r}\right)^n .$$

Corollary 3 Let a, b, c be positive real numbers and let

$$F(x) \;=\; F(x; a, b, c) \;=\; \frac{a + b + c}{a^x + b^x + c^x}\sum_{\text{cyclic}}\frac{a^x + b^x}{a + b},$$

$$E(x) \;=\; E(x; a, b, c) \;=\; \frac{1}{a^x + b^x + c^x}\sum_{\text{cyclic}}\frac{a\left(b^x + c^x\right)}{b + c}.$$

Then $F(x)$ and $E(x)$ are each decreasing on $(0, \infty)$.

Proof: We have

$$L_1(x) \;=\; \frac{\sigma(1)}{\sigma(x)}\sum_{\text{cyclic}}\frac{\sigma(x)}{b + c} - \frac{\sigma(1)}{\sigma(x)}\sum_{\text{cyclic}}\frac{b^x + c^x}{b + c} \;=\; \sum_{\text{cyclic}}\frac{a + b + c}{b + c} - F(x),$$

hence, $F(x)$ is decreasing on $(0, \infty)$ because $L_1(x)$ is increasing on $(0, \infty)$ by Theorem 2. Straightforward calculations show that $E(x) = F(x) - 2$, hence $E(x)$ is also decreasing on $(0, \infty)$. ■

We now apply the preceding results to obtain some generalizations of various problems.

Problem For any positive real numbers a, b, c, r and any positive real numbers p, q such that $q < r < p$ prove the following inequalities:

$$\frac{1}{\sigma(p)} \sum_{\text{cyclic}} \frac{a^p + b^p}{a^r + b^r} \;\leq\; \frac{3}{\sigma(r)} \;\leq\; \frac{1}{\sigma(q)} \sum_{\text{cyclic}} \frac{a^q + b^q}{a^r + b^r}\,; \tag{4}$$

$$\frac{1}{\sigma(p)} \sum_{\text{cyclic}} \frac{a^r\left(b^p + c^p\right)}{b^r + c^r} \;\leq\; 1 \;\leq\; \frac{1}{\sigma(q)} \sum_{\text{cyclic}} \frac{a^r\left(a^q + b^q\right)}{a^r + b^r}\,. \tag{5}$$

Solution: We have $F(\frac{p}{r}; a^r, b^r, c^r) \leq F(1; a^r, b^r, c^r) \leq F(\frac{q}{r}; a^r, b^r, c^r)$ by Corollary 2, and since $F(1; a^r, b^r, c^r) = 3$ the first inequality follows.

Similarly, $E(\frac{p}{r}; a^r, b^r, c^r) \leq E(1; a^r, b^r, c^r) \leq E(\frac{q}{r}; a^r, b^r, c^r)$ and since $E(1; a^r, b^r, c^r) = 1$ the second inequality follows. ■

Inequality (4) is a generalization of the inequality $\sum\limits_{\text{cyclic}} \dfrac{a^2 + b^2}{a + b} \leq \dfrac{3\sigma(2)}{\sigma(1)}$ in [2], and also a generalization of the inequality in [3].

Inequality (5) generalizes the inequality $\sum\limits_{\text{cyclic}} \dfrac{x^p(y + z)}{y^p + z^p} \geq x + y + z$, for positive x, y, z, and $p > 1$, which is Peter Woo's generalization of the inequality in [4] (see the commentary on p. 180). Furthermore, by using the rightmost relation of Inequality (5) we can obtain a generalization of the inequality $\sum\limits_{\text{cyclic}} \dfrac{a^{\lambda+1}}{b^\lambda + c^\lambda} \geq \dfrac{a + b + c}{2}$, for $\lambda \geq 0$, suggested by Walther Janous in [4] (again, see the commentary on p. 180). Namely: for any positive real numbers a, b, c, p, and q the following inequality holds

$$\sum_{\text{cyclic}} \frac{a^{p+q}}{b^p + c^p} \;\geq\; \frac{a^q + b^q + c^q}{2}\,. \tag{6}$$

Proof: The inequality $\dfrac{a^{p+q}\left(b^q + c^q\right)}{b^{p+q} + c^{p+q}} \leq \dfrac{2a^{p+q}}{b^p + c^p}$ holds since simple manipulations show that it is equivalent to $\left(b^q - c^q\right)\left(b^p - c^p\right) \geq 0$, and from inequality (5) it follows that $\sum\limits_{\text{cyclic}} \dfrac{a^{p+q}\left(b^q + c^q\right)}{b^{p+q} + c^{p+q}} \geq a^q + b^q + c^q$, hence,

$$\sum_{\text{cyclic}} \frac{a^{p+q}}{b^p + c^p} \;\geq\; \frac{1}{2} \sum_{\text{cyclic}} \frac{a^{p+q}\left(b^q + c^q\right)}{b^{p+q} + c^{p+q}} \;\geq\; \frac{a^q + b^q + c^q}{2}\,,$$

which proves inequality (6).

In [1] the inequality $\sum\limits_{\text{cyclic}} \left(\dfrac{c^2}{a^2 + b^2}\right)^n \geq \sum\limits_{\text{cyclic}} \left(\dfrac{c}{a + b}\right)^n$ was suggested. The next theorem offers a generalization.

Theorem 3 Let n be a positive integer and a, b, c be positive real numbers. Then $G(x) = G_n(x; a, b, c) = \sum_{\text{cyclic}} \left(\frac{c^x}{a^x + b^x}\right)^n$ is increasing on $(0, \infty)$.

Proof: Let $p > q > 0$ and let $A_x = \frac{a^x}{\sigma(x)}$, $B_x = \frac{b^x}{\sigma(x)}$, and $C_x = \frac{c^x}{\sigma(x)}$. Then we obtain

$$G_n(p) \geq G_n(q) \iff \sum_{\text{cyclic}} \frac{A_p^n}{(1-A_p)^n} \geq \sum_{\text{cyclic}} \frac{A_q^n}{(1-A_q)^n}$$

$$\iff \sum_{\text{cyclic}} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} A_p^{k+n} \geq \sum_{\text{cyclic}} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} A_q^{k+n}$$

$$\iff \sum_{k=1}^{\infty} \binom{k+n-1}{n-1} \sum_{\text{cyclic}} A_p^{k+n} \geq \sum_{k=1}^{\infty} \binom{k+n-1}{n-1} \sum_{\text{cyclic}} A_q^{k+n}$$

$$\iff \sum_{k=1}^{\infty} \binom{k+n-1}{n-1} \frac{\sigma\big((k+n)p\big)}{\sigma^{k+n}(p)} \geq \sum_{k=1}^{\infty} \binom{k+n-1}{n-1} \frac{\sigma\big((k+n)q\big)}{\sigma^{k+n}(q)},$$

and the last inequality above holds termwise by the result of Corollary 1. ∎

By applying the result of Theorem 3 to the terms of an infinite series we obtain the following corollary.

Corollary 4 Let $h(t) = \sum_{n=0}^{\infty} h_n t^n$, where each h_n is nonnegative and the series converges for $t \geq 0$. Then for any positive real numbers a, b, c the function $G_h(x; a, b, c) = \sum_{\text{cyclic}} h\left(\frac{c^x}{a^x + b^x}\right)$ is increasing in x on $(0, \infty)$.

References

[1] Razvan Satnianu, Problem 11080, *American Mathematical Monthly*, Vol. 111, No. 4.

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[3] Arkady Alt, Problem 3300, ***CRUX Mathematicorum with Mathematical Mayhem***, Vol. 33, No. 8 (2007) p. 489.

[4] Sefket Arslanagic, Problem 2927*, ***CRUX Mathematicorum with Mathematical Mayhem***, Vol. 30, No. 3 (2004) p. 172; solution in Vol. 31, No. 3 (2005) pp. 179-180.

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 May 2010**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

3464. Correction. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle with $\angle A = 90^\circ$ and H be the foot of the altitude from A. let J be the point on the hypotenuse BC such that $CJ = HB$ and let K, L be the projections of J onto AB, AC, respectively. Prove that

$$\mathcal{M}\left(-\frac{2}{3}; AK, AL\right) \;\le\; \frac{1}{2}\mathcal{M}\left(-2; AB, AC\right)\ ,$$

where $\mathcal{M}(\alpha; x, y) = \left(\dfrac{x^\alpha + y^\alpha}{2}\right)^{1/\alpha}$.

3475. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be an equilateral triangle with side length a, and let P be a point on the line BC such that $AP = 2x > a$. Let M be the midpoint of AP. If $\dfrac{BM}{x} = \dfrac{BP}{a} = \alpha$ and $\dfrac{CM}{x} = \dfrac{CP}{a} = \beta$, find x, α, and β.

3476. *Proposed by Michel Bataille, Rouen, France.*

Let ℓ be a line and O be a point not on ℓ. Find the locus of the vertices of the rectangular hyperbolas centred at O and tangent to ℓ. (A hyperbola is rectangular if its asymptotes are perpendicular.)

3477. *Proposed by Michel Bataille, Rouen, France.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2y^2\big(f(x+y) - f(x) - f(y)\big) \;=\; 3(x+y)f(x)f(y)$$

for all real numbers x and y.

3478. *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a and b be positive real numbers. Prove that

$$\frac{a}{b} \;+\; \frac{b}{a} \;+\; \sqrt{1 + \frac{2ab}{a^2+b^2}} \;\ge\; 2 + \sqrt{2}\,.$$

3479. *Proposed by Jonathan Schneider, student, University of Toronto Schools, Toronto, ON.*

The real numbers x, y, and z satisfy the system of equations

$$\begin{aligned} x^2 - x &= yz + 1\,, \\ y^2 - y &= xz + 1\,, \\ z^2 - z &= xy + 1\,. \end{aligned}$$

Find all solutions (x, y, z) of the system and determine all possible values of $xy + yz + zx + x + y + z$ where (x, y, z) is a solution of the system.

3480. *Proposed by Bianca-Teodora Iordache, Carol I National College, Craiova, Romania.*

Let a_1, a_2, $\dots$, a_n $(n \geq 3)$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n \;\geq\; a_1 a_2 \cdots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\,.$$

Prove that $a_1^n + a_2^n + \cdots + a_n^n \geq a_1^{n-1} a_2^{n-1} \cdots a_n^{n-1}$. Find a necessary and sufficient condition for equality to hold.

3481. *Proposed by Joe Howard, Portales, NM, USA.*

Let $\triangle ABC$ have at most one angle exceeding $\frac{\pi}{3}$. If $\triangle ABC$ has area F and side lengths a, b, and c, prove that

$$(ab + bc + ca)^2 \;\geq\; 4\sqrt{3} \cdot F\left(a^2 + b^2 + c^2\right)\,.$$

3482. *Proposed by José Luis Díaz-Barrero and Josep Rubió-Massegú, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let $a_n \neq 0$ and $p(z) = \sum\limits_{k=0}^{n} a_k z^k$ be a polynomial with complex coefficients and zeros z_1, z_2, $\dots$, z_n, such that $|z_k| < R$ for each k. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{R^2 - |z_k|^2}} \;\geq\; \frac{2}{R^2}\left|\frac{a_{n-1}}{a_n}\right|\,.$$

When does equality occur?

3483. *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let $n \geq 3$ be an integer and let x_1, x_2, $\dots$, x_n be positive real numbers. Prove that

$$\left(\frac{x_1}{x_2}\right)^{n-2} + \left(\frac{x_2}{x_3}\right)^{n-2} + \cdots + \left(\frac{x_n}{x_1}\right)^{n-2} \;\geq\; \frac{x_1 + x_2 + \cdots + x_n}{\sqrt[n]{x_1 x_2 \cdots x_n}}\,.$$

3484★. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let N be a positive integer with decimal expansion $N = a_1 a_2 \ldots a_r$, where r is the number of decimal digits and $0 \le a_i \le 9$ for each i, except for a_1, which must be positive. Let $s(N) = a_1 + a_2 + \cdots + a_r$. Find all pairs (N, p) of positive integers such that $\big(s(N)\big)^p = s\big(N^p\big)$.

3485. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let x, y, z be positive real numbers in the interval $[0, 1]$. Prove that

$$\frac{x}{y+z+1} + \frac{y}{x+z+1} + \frac{z}{x+y+1} + (1-x)(1-y)(1-z) \;\le\; 1\,.$$

3486. *Proposed by Pham Huu Duc, Ballajura, Australia.*

Let a, b, and c be positive real numbers. Prove that

$$\frac{bc}{a^2+bc} + \frac{ca}{b^2+ca} + \frac{ab}{c^2+ab} \;\le\; \frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\,.$$

3487★. *Proposed by Neven Jurič, Zagreb, Croatia.*

Does the following hold for every positive integer n?

$$\sum_{k=0}^{n-1} (-1)^k \frac{1}{2n-2k-1}\binom{2n-1}{k} \;=\; (-1)^{n-1}\frac{16^n}{8n\binom{2n}{n}}\,.$$

. . . .

3464. Correction. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle avec $\angle A = 90^\circ$ et H le pied de la hauteur abaissée de A. Soit J le point sur l'hypoténuse BC tel que $CJ = HB$ et soit K, L les projections respectives de J sur AB, AC. Montrer que

$$\mathcal{M}\left(-\frac{2}{3}; AK, AL\right) \;\le\; \frac{1}{2}\mathcal{M}\left(-2; AB, AC\right)\,,$$

où $\mathcal{M}(\alpha; x, y) = \left(\dfrac{x^\alpha + y^\alpha}{2}\right)^{1/\alpha}$.

3475. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle équilatéral de côté a et soit P un point sur la droite BC tel que $AP = 2x > a$. Soit M le point milieu de AP. Si $\dfrac{BM}{x} = \dfrac{BP}{a} = \alpha$ et $\dfrac{CM}{x} = \dfrac{CP}{a} = \beta$, trouver x, α et β.

3476. *Proposé par Michel Bataille, Rouen, France.*

Soit ℓ une droite et O un point non sur ℓ. Trouver le lieu des sommets des hyperboles rectangulaires centrées en O et tangentes à ℓ. (Une hyperbole est rectangulaire si ses asymptotes sont perpendiculaires.)

3477. *Proposé par Michel Bataille, Rouen, France.*

Trouver toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que

$$x^2y^2\big(f(x+y) - f(x) - f(y)\big) \;=\; 3(x+y)f(x)f(y)$$

pour tous les nombres réels x et y.

3478. *Proposé par Cao Minh Quang, Collège Nguyen Binh Khiem, Vinh Long, Vietnam.*

Soit a et b deux nombres réels positifs. Montrer que

$$\frac{a}{b} + \frac{b}{a} + \sqrt{1 + \frac{2ab}{a^2+b^2}} \;\geq\; 2 + \sqrt{2}\,.$$

3479. *Proposé par Jonathan Schneider, étudiant, University of Toronto Schools, Toronto, ON.*

Les nombres réels x, y et z satisfont le système d'équations

$$\begin{aligned} x^2 - x &= yz + 1\,, \\ y^2 - y &= xz + 1\,, \\ z^2 - z &= xy + 1\,. \end{aligned}$$

Trouver toutes les solutions (x, y, z) du système et déterminer toutes les valeurs possibles de $xy + yz + zx + x + y + z$, où (x, y, z) est une solution du système.

3480. *Proposé par Bianca-Teodora Iordache, Collège National Carol Ier, Craiova, Roumanie.*

Soit a_1, a_2, ..., a_n $(n \geq 3)$ des nombres réels positifs tels que

$$a_1 + a_2 + \cdots + a_n \;\geq\; a_1a_2\cdots a_n\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\,.$$

Montrer que $a_1^n + a_2^n + \cdots + a_n^n \geq a_1^{n-1}a_2^{n-1}\cdots a_n^{n-1}$. Trouver une condition nécessaire et suffisante pour qu'on ait l'égalité.

3481. *Proposé par Joe Howard, Portales, NM, É-U.*

Soit ABC un triangle avec au plus un angle excédant $\frac{\pi}{3}$. Si ABC a une aire de F et des côtés de longueur a, b et c, montrer que

$$(ab + bc + ca)^2 \;\geq\; 4\sqrt{3}\cdot F\big(a^2 + b^2 + c^2\big)\,.$$

3482. *Proposé par José Luis Díaz-Barrero et Josep Rubió-Massegú, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit $a_n \neq 0$ et $p(z) = \sum\limits_{k=0}^{n} a_k z^k$ un polynôme à coefficients complexes ayant des racines $z_1, z_2, \dots, z_n$ telles que $|z_k| < R$ pour chaque k. Montrer que

$$\sum_{k=1}^{n} \frac{1}{\sqrt{R^2 - |z_k|^2}} \;\geq\; \frac{2}{R^2}\left|\frac{a_{n-1}}{a_n}\right| .$$

Quand a-t-on l'égalité ?

3483. *Proposé par Cao Minh Quang, Collège Nguyen Binh Khiem, Vinh Long, Vietnam.*

Soit un entier $n \geq 3$ et soit $x_1, x_2, \dots, x_n$ n nombres réels positifs. Montrer que

$$\left(\frac{x_1}{x_2}\right)^{n-2} + \left(\frac{x_2}{x_3}\right)^{n-2} + \cdots + \left(\frac{x_n}{x_1}\right)^{n-2} \;\geq\; \frac{x_1 + x_2 + \cdots + x_n}{\sqrt[n]{x_1 x_2 \cdots x_n}} .$$

3484★. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit N un entier positif dont l'écriture décimale est $N = a_1 a_2 \dots a_r$, où r est le nombre de chiffres décimaux, tous compris entre 0 et 9 sauf a_1 qui doit être positif. Soit aussi $s(N) = a_1 + a_2 + \cdots + a_r$. Trouver toutes les paires (N, p) d'entiers positifs telles que $\big(s(N)\big)^p = s\big(N^p\big)$.

3485. *Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.*

Soit x, y, z des nombres réels positifs dans l'intervalle $[0, 1]$. Montrer que

$$\frac{x}{y+z+1} + \frac{y}{x+z+1} + \frac{z}{x+y+1} + (1-x)(1-y)(1-z) \;\leq\; 1 .$$

3486. *Proposé par Pham Huu Duc, Ballajura, Australie.*

Soit a, b et c trois nombres réels positifs. Montrer que

$$\frac{bc}{a^2 + bc} + \frac{ca}{b^2 + ca} + \frac{ab}{c^2 + ab} \;\leq\; \frac{1}{2}\sqrt[3]{3(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} .$$

3487★. *Proposé par Neven Jurič, Zagreb, Croatie.*

L'égalité suivante est-elle valable pour tout entier positif n ?

$$\sum_{k=0}^{n-1} (-1)^k \frac{1}{2n-2k-1}\binom{2n-1}{k} \;=\; (-1)^{n-1}\frac{16^n}{8n\binom{2n}{n}} .$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We belatedly acknowledge a correct solution to #3340 by "Solver X", dedicated to the memory of Jim Totten, which we had previously classified as incorrect. Our apologies.

3376. [2008 : 430, 432] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

The vertices of quadrilateral $ABCD$ lie on a circle. Let K, L, M, and N be the incentres of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$, respectively. Show that quadrilateral $KLMN$ is a rectangle.

Comment by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

This problem has already appeared in Crux. The proof of the result and the proof of its converse was published together with comments and references in [1980 : 226-230].

Solutions, comments, and other references were sent by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Geupel provides the link `www.mathlinks.ro/viewtopic.php?t=137589`. *Janous provides the link* `http://www.gogeometry.com/problem/p035_incenter_cyclic_quadrilateral`. *Konečný and Schlosberg found a solution at* `www.cut-the-knot.org/Curriculum/Geometry/CyclicQuadrilateral.shtml` *Schlosberg also gives two other links,* `forumgeom.fau.edu/FG2002volume2/FG200223.ps` *and* `mathworld.wolfram.com/CyclicQuadrilateral.html`.

3377. [2008 : 430, 432] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let ABC be a triangle with $\angle B = 2\angle C$. The interior bisector of $\angle BAC$ meets BC at D. Let M and N be the midpoints of AC and BD, respectively. Suppose that A, M, D, and N are concyclic. Prove that $\angle BAC = 72^\circ$.

Solution by Michel Bataille, Rouen, France.

We will use the familiar notations for the elements of the triangle ABC. First, we note that from $\angle B = 2\angle C$, we have

$$b^2 \;=\; c(c+a) \qquad (1)$$

(this has been proven in the April 2006 issue of this journal, [2006 : 159]).

It then suffices to prove that $a = b$, for then we will have $B = A$, and then

$$A = \frac{2}{5}\left(A + A + \frac{A}{2}\right) = \frac{2}{5}(A + B + C) = 72^\circ .$$

From $\frac{DB}{c} = \frac{DC}{b} = \frac{a}{b+c}$, we obtain $DB = \frac{ac}{b+c}$ and $DC = \frac{ab}{b+c}$, so that

$$CN = \frac{ab}{b+c} + \frac{ac}{2(b+c)} = \frac{a(2b+c)}{2(b+c)} .$$

Since A, M, D, and N are concyclic, we have $CM \cdot CA = CD \cdot CN$, or, after a simple calculation,

$$c^2 b = c\left(a^2 - 2b^2\right) + b\left(2a^2 - b^2\right) . \qquad (2)$$

The relation (1), rewritten as $c^2 b = b^3 - abc$, together with (2) yields

$$c(a-b)(2b+a) = 2b(b-a)(b+a) .$$

Now, if $a \neq b$, then $c < 0$, which is impossible. Therefore, $a = b$, which completes the proof.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; SOUTHEAST MISSOURI STATE UNIVERSITY MATH CLUB; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ALBERT STADLER, Herrliberg, Switzerland; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Amengual Covas noted that the relation (1) *had already appeared in several other issues of* ***CRUX****, namely,* [1976 : 74], [1984 : 278], *and* [1996 : 265-267].

3378. [2008 : 430, 432] *Proposed by Mihály Bencze, Brasov, Romania.*

Let x, y, and z be positive real numbers. Prove that

$$\sum_{\text{cyclic}} \frac{xy}{xy + x + y} \leq \sum_{\text{cyclic}} \frac{x}{2x + z} .$$

Counterexample by George Apostolopoulos, Messolonghi, Greece.

The inequality is false in general. For example, if $x = 2$ and $y = z = 1$, then the inequality becomes $\frac{2}{5} + \frac{1}{3} + \frac{2}{5} \leq \frac{2}{5} + \frac{1}{4} + \frac{1}{3}$, or $\frac{2}{5} < \frac{1}{4}$, which is clearly false.

Also disproved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State

University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; STAN WAGON, Macalester College, St. Paul, MN, USA; and TITU ZVONARU, Comănești, Romania.

Let $f(x,y,z) = \sum_{\text{cyclic}} \frac{x}{2x+z} - \sum_{\text{cyclic}} \frac{xy}{xy+x+y}$. *Then Curtis showed that* $f(x,x,x) < 0$ *for all* $x > 1$. *Perfetti showed that, in general,* $f(x,y,z) < 0$ *if* x, y, z *all lie in* $(1,\infty)$, *while* $f(x,y,z) \geq 0$ *if* x, y, z *all lie in* $(0,1]$.

Janous reports that a similar inequality, $\sum_{\text{cyclic}} \frac{xy}{xy+x^2+y^2} \leq \sum_{\text{cyclic}} \frac{x}{2x+z}$, *where* x, y, z *are positive, was problem 4 of the 2009 Mediterranean Mathematics Competition.*

3379. [2008 : 430, 433] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a_1, a_2, ..., a_n be positive real numbers. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{a_i + (n-1)a_{i+1}} \geq 1,$$

where the subscripts are taken modulo n.

Solution by Michel Bataille, Rouen, France.

Set $x_i = \frac{(n-1)a_{i+1}}{a_i}$ for each i; the problem is then to prove that

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \geq 1 \qquad (1)$$

subject to the constraint $x_1 x_2 \cdots x_n = (n-1)^n$.

Let $y_i = \frac{1}{1+x_i}$ for each i, so that $x_i = \frac{1-y_i}{y_i}$.

Suppose on the contrary that $\sum_{i=1}^{n} y_i < 1$. Then $1 - \sum_{i=1}^{n} y_i > 0$, and hence for each i we have by the AM–GM Inequality that

$$1 - y_i > \sum_{j \neq i} y_j \geq (n-1)\left(\prod_{j\neq i} y_j\right)^{\frac{1}{n-1}}. \qquad (2)$$

Multiplying across the inequalities in (2) over all i, we then obtain

$$\prod_{i=1}^{n}(1-y_i) > (n-1)^n \prod_{i=1}^{n} y_i,$$

or $\prod_{i=1}^{n}\left(\frac{1-y_i}{y_i}\right) > (n-1)^n$, which violates the constraint on x_1, x_2, ..., x_n.

This contradiction establishes (1), and we are done.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; CRISTINEL MORTICI, Valahia University of Târgoviște, Romania; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

For positive α Janous proved the generalization

$$\sum_{i=1}^{n} \frac{a_i}{a_i + \alpha a_{i+1}} \geq \min\left\{\frac{n}{1+\alpha}, 1\right\}.$$

3380. [2008 : 430, 433] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, b, c, x, y, and z be real numbers. Show that

$$\frac{(a^2+x^2)(b^2+y^2)}{(c^2+z^2)(a-b)^2} + \frac{(b^2+y^2)(c^2+z^2)}{(a^2+x^2)(b-c)^2} + \frac{(c^2+z^2)(a^2+x^2)}{(b^2+y^2)(c-a)^2}$$
$$\geq \frac{a^2+x^2}{|(a-b)(a-c)|} + \frac{b^2+y^2}{|(b-a)(b-c)|} + \frac{c^2+z^2}{|(c-a)(c-b)|}.$$

Similar solutions by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France.

Let

$$u = \sqrt{\frac{(a^2+x^2)(b^2+y^2)}{(c^2+z^2)(a-b)^2}}; \qquad v = \sqrt{\frac{(b^2+y^2)(c^2+z^2)}{(a^2+x^2)(b-c)^2}};$$
$$w = \sqrt{\frac{(c^2+z^2)(a^2+x^2)}{(b^2+y^2)(c-a)^2}}.$$

Then

$$uv = \sqrt{\frac{(b^2+y^2)^2}{(a-b)^2(b-c)^2}} = \frac{b^2+y^2}{|(a-b)(b-c)|},$$

and similarly

$$uw = \frac{a^2+x^2}{|(a-b)(c-a)|}; \qquad vw = \frac{c^2+z^2}{|(b-c)(c-a)|}.$$

The original inequality now follows from the well-known inequality

$$u^2 + v^2 + w^2 \geq uv + uw + vw.$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, *Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; TITU ZVONARU, Comănești, Romania; and the proposer.*

3381★. [2008 : 431, 433] *Proposed by Shi Changwei, Xi`an City, Shaan Xi Province, China.*

Let n be a positive integer. Prove that

(a) $\left(1-\frac{1}{6}\right)\left(1-\frac{1}{6^2}\right)\cdots\left(1-\frac{1}{6^n}\right) > \frac{4}{5}$;

(b) Let $a_n = aq^n$, where $0 < a < 1$ and $0 < q < 1$. Evaluate

$$\lim_{n\to\infty}\prod_{i=1}^{n}(1-a_i)\,.$$

Solution to part (a) *by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

We have $(1-aq)\left(1-aq^2\right) > 1-a\left(q+q^2\right)$. In general, if we know that $\prod\limits_{k=1}^{n}\left(1-aq^k\right) > 1-\sum\limits_{k=1}^{n} aq^k$ holds for some $n \geq 2$, then

$$\begin{aligned}\prod_{k=1}^{n+1}\left(1-aq^k\right) &> \left(1-aq^{n+1}\right)\left(1-\sum_{k=1}^{n} aq^k\right)\\ &= 1 - \sum_{k=1}^{n+1} aq^k + aq^{n+1}\left(\sum_{k=1}^{n} aq^k\right)\\ &> 1 - \sum_{k=1}^{n+1} aq^k\,.\end{aligned}$$

By induction, the inequality holds for each $n \geq 2$, and furthermore we have $\prod\limits_{k=1}^{n}\left(1-aq^k\right) > 1-\sum\limits_{k=1}^{n} aq^k > 1-\dfrac{aq}{(1-q)}$. Taking $a=1$ and $q=\dfrac{1}{6}$ yields $\left(1-\frac{1}{6}\right)\left(1-\frac{1}{6^2}\right)\cdots\left(1-\frac{1}{6^n}\right) > 1-\frac{1}{5} = \frac{4}{5}$.

Also solved by ARKADY ALT, San Jose, CA, USA (*part* (a))*; GEORGE APOSTOLOPOULOS, Messolonghi, Greece* (*part* (a))*; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA* (*part* (a))*; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ALBERT STADLER, Herrliberg, Switzerland; and STAN WAGON, Macalester College, St. Paul, MN, USA. There was one incorrect solution to part* (b) *submitted.*

Although it is a fundamental problem to evaluate the limit in part (b)*, there appears to be no simple way to tame this infinite product.*

Bracken notes that the product is related to Ramanujan's q-extension of the Gamma function, the customary notations being $(a;q)_k = (1-a)(1-aq)\cdots\left(1-aq^{k-1}\right)$ and $(a;q)_\infty = \prod\limits_{k=0}^{\infty}\left(1-aq^k\right)$. The q-binomial theorem implies that for $|x|<1$, $|q|<1$ we have $\frac{1}{(a;q)_\infty} = \sum\limits_{n=0}^{\infty}\frac{x^n}{(q;q)_n}$, which leads to an expression for the required product.

Geupel gives the expansion

$$\prod_{k=1}^{\infty}\left(1-q^k\right) = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt[24]{q}}\cdot\vartheta_2\left(\frac{\pi}{6},\sqrt[6]{q}\right),$$

where ϑ_2 is a Jacobi theta function. For further information he refers to the online survey `http://mathworld.wolfram.com/q-PochhammerSymbol.html`.

Stadler provided six different expansions related to the required product, one involving the Dedekind eta function and the others due to Euler and Jacobi. He refers the interested reader to Marvin I. Knopp, Modular Functions in Analytic Number Theory (2^{nd} ed.), *Chelsea, New York (2003).*

Wagon also refers to the Wolfram website, as well as B. Gordon and R.J. McIntosh, "Some Eighth Order Mock Theta Functions", J. London Math. Soc. **62**, *pp. 321-335 (2000). There is given the asymptotic estimate $(q;q)_\infty = \sqrt{\frac{\pi}{t}}\exp\left(\frac{t}{12}-\frac{\pi^2}{12t}\right)+o(1)$; $t=-\frac{1}{2}\ln q$, which yields a value for $(1/6;1/6)_\infty$ that is within $2\cdot 10^{-10}$ of the true value.*

3382. [2008 : 431, 433] *Proposed by José Luis Díaz-Barrero and Josep Rubió-Massegú, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a positive integer. Prove that

$$\sin\left(\frac{P_{n+2}}{4P_nP_{n+1}}\right) + \cos\left(\frac{P_{n+2}}{4P_nP_{n+1}}\right) < \frac{3}{2}\sec\left(\frac{3P_n+2P_{n-1}}{4P_nP_{n+1}}\right),$$

where P_n is the n^{th} Pell number, which is defined by $P_0=0$, $P_1=1$, and $P_n = 2P_{n-1}+P_{n-2}$ for $n\ge 2$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

For all real numbers x and y with $0<y<\frac{\pi}{2}$, we have

$$\sin x+\cos x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right) \le \sqrt{2} < \frac{3}{2} \le \frac{3}{2}\sec y.$$

Setting $x = \frac{P_{n+2}}{4P_nP_{n+1}}$ and $y = \frac{3P_n+2P_{n-1}}{4P_nP_{n+1}}$ gives the desired result, if we can show that $0<y<\frac{\pi}{2}$.

Because the Pell numbers form a strictly increasing sequence, we obtain for positive n that

$$0 < \frac{3P_n+2P_{n-1}}{4P_nP_{n+1}} < \frac{5P_n}{4P_n^2} \le \frac{5}{4} < \frac{\pi}{2},$$

which completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposers.

It is clear from the featured solution that on the right side of the inequality $\frac{3}{2}$ can be replaced by the smaller coefficient $\sqrt{2}$.

3383. [2008 : 431, 433] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle with $\angle BAC \neq 90^\circ$, let O be its circumcentre and let M be the midpoint of BC. Let P be a point on the ray MA such that $\angle BPC = 180^\circ - \angle BAC$. Let BP meet AC at E and let CP meet AB at F. If D is the projection of the midpoint of EF onto BC, show that

(a) AD is a symmedian of $\triangle ABC$;

(b) O, P, and the orthocentre of $\triangle EDF$ are collinear.

A composite of solutions by Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India and the proposer.

We first show that there is a unique point P on the ray MA such that $\angle BPC = 180^\circ - \angle BAC$, and P can be constructed by extending the median AM to where it meets the circumcircle Γ of $\triangle ABC$ at P'. *Claim*: The line through B parallel to $P'C$ meets AM at P and AC at E. To see this, note that $\triangle PBM \cong \triangle P'CM$ ($BM = MC$ and corresponding angles are equal), whence $PBP'C$ is a parallelogram. It follows that $\angle BPC = \angle CP'B$. But P', A lie on opposite arcs BC of Γ, so $\angle BPC = \angle CP'B = 180^\circ - \angle BAC$, whence P is the unique point on the ray MA that forms the required angle. Moreover, since $\angle BAC \neq 90^\circ$, P is different from A. The homothety with centre A that takes P' to P will take C to E, B to F, and Γ to the circumcircle Γ' of the quadrilateral $AFPE$. Note that EF is parallel to BC; its midpoint, call it N, lies on AP.

We turn now to part (a). Since AP and FE intersect at N, by the classical construction N is the pole of the line BC with respect to Γ' (because B and C are the diagonal points other than N of quadrilateral $AFPE$). The polar of D, therefore, passes through N and, since DN passes through the centre of Γ' and is perpendicular to EF, this polar is actually EF. As a result, DE and DF are tangent to Γ' at E and F. It follows that AD is the symmedian from A in $\triangle EAF$. (See, for example, Roger A. Johnson, *Advanced Euclidean Geometry*, (Dover, 2007), page 215, no. 347, or Nathan Altshiller Court, *College Geometry*, (Dover, 2007), page 248, Theorem 560.) The result in (a) follows since $\triangle BAC$ and $\triangle FAE$ are homothetic. For part (b) note that the homothety that takes the circle BAC (namely Γ) to FAE (namely Γ') will take OB to the radius of Γ' through F; since DF is the tangent at F, its preimage is tangent to Γ at B, and is therefore perpendicular to OB. We conclude that $OB \perp DF$. Similarly, $OC \perp DE$. Let H' denote the orthocentre of $\triangle EDF$. The homothety with centre P that takes B to E will take C to F. Since $OB \| H'E$ and $OC \| H'F$, this homothety must take O to H', whence H', P, and O are collinear, which completes the proof of part (b).

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

All the submitted solutions except Bataille's were based on the property that AD is a symmedian of $\triangle ABC$ if and only if D divides the side BC in the ratio $c^2 : b^2$ (Court, page 248, Theorem 561).

3384. [2008 : 431, 434] *Proposed by Michel Bataille, Rouen, France.*

Show that, for any real number x,

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n-1} k \cdot \left\lfloor x + \frac{n-k-1}{n} \right\rfloor = \frac{\lfloor x \rfloor + \{x\}^2}{2},$$

where $\lfloor a \rfloor$ is the integer part of the real number a and $\{a\} = a - \lfloor a \rfloor$.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

Let $T_n = \sum\limits_{k=1}^{n-1} k \cdot \left\lfloor x + \frac{n-k-1}{n} \right\rfloor$. Given an integer $n \geq 2$, let m be the unique integer such that $\frac{m}{n} \leq \{x\} < \frac{m+1}{n}$. Note that $m = m(n)$ is a function of n, and that $\frac{m}{n} \to \{x\}$ as $n \to \infty$ since $\left|\{x\} - \frac{m}{n}\right| < \frac{1}{n}$. Now, the first $m-1$ terms of T_n are $\big(\lfloor x \rfloor + 1\big)\big(1+2+\cdots+(m-1)\big)$, and the final $n-m$ terms of T_n are $\lfloor x \rfloor \big(m + (m+1) + \cdots + (n-1)\big)$. Collect all terms involving $\lfloor x \rfloor$ and close the sums to obtain $\frac{1}{n^2} T_n = \frac{1}{2} \lfloor x \rfloor \left(\frac{n-1}{n}\right) + \frac{1}{2}\left(\frac{m}{n}\right)\left(\frac{m-1}{n}\right)$; it then follows that $\frac{1}{n^2} T_n \to \frac{1}{2}\left(\lfloor x \rfloor + \{x\}^2\right)$ as $n \to \infty$.

Also solved by ARKADY ALT, San Jose, CA, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

The Missouri State University Problem Solving Group reduced the problem to calculating $\int_0^1 y \lfloor x + 1 - y \rfloor\, dy$ by noting that the required limit is the limit of a Riemann sum for the integrand on $[0, 1]$ plus a term that vanishes as $n \to \infty$.

3385. [2008 : 431, 434] *Proposed by Michel Bataille, Rouen, France.*

Let p_1, p_2, ..., p_6 be primes with $p_{k+1} = 2p_k + 1$ for $k = 1, 2, \ldots, 5$. Show that

$$\sum_{1 \le i < j \le 6} p_i p_j$$

is divisible by 15.

A composite of similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Cristinel Mortici, Valahia University of Târgovişte, Romania; and Titu Zvonaru, Comăneşti, Romania.

If $p_1 = 3$, then $p_2 = 7$ and $p_3 = 15$, whence p_3 is not prime and, therefore, our sequence cannot begin with the prime $p_1 = 3$. Nor can it begin with $p_1 \equiv 1 \pmod 3$, otherwise we would have $p_2 \equiv 0 \pmod 3$, so that p_2 could not be prime. We conclude that necessarily, $p_1 \equiv -1 \pmod 3$, and hence that $p_i \equiv -1 \pmod 3$ for each p_i. The resulting sum satisfies

$$\sum_{1 \le i < j \le 6} p_i p_j \equiv \sum_{1 \le i < j \le 6} (-1)^2 \equiv \binom{6}{2} \equiv 0 \pmod 3 .$$

Similarly, working modulo 5, we see that

(a) If $p_1 = 5$, then $p_5 \equiv 0 \pmod 5$ and is not prime.
(b) If $p_1 \equiv 1 \pmod 5$, then $p_4 \equiv 0 \pmod 5$ and is not prime.
(c) If $p_1 \equiv 2 \pmod 5$, then $p_6 \equiv 0 \pmod 5$ and is not prime.
(d) If $p_1 \equiv 3 \pmod 5$, then $p_3 \equiv 0 \pmod 5$ and is not prime.

Once again, the only possible value for p_i is $-1 \pmod 5$, whence

$$\sum_{1 \le i < j \le 6} p_i p_j \equiv \sum_{1 \le i < j \le 6} (-1)^2 \equiv \binom{6}{2} \equiv 0 \pmod 5 .$$

We have seen that the sum is divisible by 3 and by 5, and thus by 15 as claimed. Finally, we note that the result is not vacuously true: 89, 179, 359, 719, 1439, 2879 is an example of such a sequence (and is easily seen to be the smallest example).

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, NB; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Manes points out that a prime p is called a Sophie Germain prime *if $2p + 1$ is also a prime; moreover, a sequence of $n - 1$ Sophie Germain primes, p, $2p + 1$, $2(2p + 1) + 1$, ..., that cannot be extended in either direction (that is, the first prime is not of the form $2q + 1$ for q a prime, while the final prime of the sequence is not a Sophie Germain prime) is called a* Cunningham chain of the first kind of length n*. Aside from the Cunningham chain of length 5 that begins with 2 (namely, 2, 5, 11, 23, 47), the final digit of any prime in a Cunningham chain of length four or greater must be a 9 (because the final digit cycles 1, 3, 7, 5, ...). According* *to the Cunningham Chain Records web page, the longest known Cunningham chain of the first kind has length* 17.

On a related note, compare Problem 10 on the Mathematical Competition Baltic Way 2004 [2008 : 212; 2009 : 153] *where one is asked to prove a result implying that a Cunningham chain can never be infinite.*

3386. [2008 : 432, 434] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Evaluate the integral

$$\int_0^\infty e^{-x}\left(\int_0^x \frac{e^{-t}-1}{t}\,dt\right)\ln x\,dx\,.$$

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Gerhard Kirchner, University of Innsbruck, Innsbruck, Austria.

We will use the fact that $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n} = -\ln 2$ and $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$. Also, from the product representation of the Gamma function, we have

$$\begin{aligned}
\frac{1}{\Gamma(x)} &= xe^{\gamma x}\prod_{k=1}^{\infty}\left(1+\frac{x}{k}\right)e^{-x/k} \\
\implies\quad -\frac{\Gamma'(x)}{\Gamma(x)} &= \frac{1}{x}+\gamma+\sum_{k=1}^{\infty}\left(\frac{1}{x+k}-\frac{1}{k}\right) \\
\implies\quad -\frac{\Gamma'(n+1)}{\Gamma(n+1)} &= \frac{1}{n+1}+\gamma+\sum_{k=1}^{\infty}\left(\frac{1}{n+1+k}-\frac{1}{k}\right) \\
&= \gamma-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\,,
\end{aligned}$$

where γ is Euler's constant. [*Ed. For properties of the Gamma function see* http://en.wikipedia.org/wiki/Gamma_function.]

We now compute

$$\begin{aligned}
&\int_0^\infty e^{-x}\left(\int_0^x \frac{e^{-t}-1}{t}\,dt\right)\ln x\,dx \\
= &\int_0^\infty e^{-x}\left(\int_0^x \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}t^{n-1}\,dt\right)\ln x\,dx \\
= &\int_0^\infty e^{-x}\ln x\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{x^n}{n}\,dx \;=\; \sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot n!}\int_0^\infty e^{-x}x^n\ln x\,dx \\
= &\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cdot\frac{\Gamma'(n+1)}{\Gamma(n+1)} \;=\; \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(-\gamma+\sum_{k=1}^{n}\frac{1}{k}\right) \\
= &\,\gamma\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sum_{k=1}^{n}\frac{1}{k}
\end{aligned}$$

$$
\begin{aligned}
&= \gamma \ln 2 + \int_0^1 \int_0^1 \sum_{n=1}^{\infty} (-1)^n x^{n-1} \sum_{k=1}^{n} y^{k-1}\, dxdy \\
&= \gamma \ln 2 + \int_0^1 \int_0^1 \sum_{n=1}^{\infty} (-1)^n x^{n-1} \frac{y^n - 1}{y-1}\, dxdy \\
&= \gamma \ln 2 + \int_0^1 \int_0^1 \frac{\dfrac{-y}{1+xy} + \dfrac{1}{1+x}}{y-1}\, dxdy \\
&= \gamma \ln 2 - \int_0^1 \left(\int_0^1 \frac{1}{(1+xy)(1+x)}\, dy \right) dx \;=\; \gamma \ln 2 - \int_0^1 \frac{\ln(1+x)}{x(1+x)}\, dx \\
&= \gamma \ln 2 - \int_0^1 \ln(1+x) \left(\frac{1}{x} - \frac{1}{1+x} \right) dx \\
&= \gamma \ln 2 - \int_0^1 \frac{\ln(1+x)}{x}\, dx + \frac{1}{2} \ln^2 2 \\
&= \gamma \ln 2 + \frac{1}{2} \ln^2 2 + \sum_{n=1}^{\infty} \int_0^1 \frac{(-1)^n}{n} x^{n-1}\, dx \\
&= \gamma \ln 2 + \frac{1}{2} \ln^2 2 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \;=\; \gamma \ln 2 + \frac{1}{2} \ln^2 2 - \frac{\pi^2}{12}\,.
\end{aligned}
$$

Also solved by KEE-WAI LAU, Hong Kong, China; DUNG NGUYEN MANH, High School of HUS, Hanoi, Vietnam; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; MOUBINOOL OMARJEE, Paris, France; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; NGUYEN VAN VINH, Belarusian State University, Minsk, Belarus; and the proposer. There were two incomplete solutions submitted.

3387. [2008 : 432, 434] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $k > l \geq 0$ be fixed integers. Find

$$\lim_{x\to\infty} 2^x \left(\zeta(x+k)^{\zeta(x+k)} - \zeta(x+l)^{\zeta(x+l)} \right) ,$$

where ζ is the Riemann zeta function.

Similar solutions by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy and Albert Stadler, Herrliberg, Switzerland.

For real-valued functions $f(x)$ and $g(x)$ defined on the interval $(1, \infty)$ the notation $f(x) = O\big(g(x)\big)$ will mean that there exists an $x_0 > 1$ and a positive constant C such that $|f(x)| \leq C|g(x)|$ whenever $x \geq x_0$.

We have $\zeta(x) = 1 + 2^{-x} + 3^{-x} + \sum\limits_{k=4}^{\infty} k^{-x}$ and

$$\frac{1}{(x-1)4^{x-1}} = \int_4^\infty \frac{ds}{s^x} \le \sum_{k=4}^{\infty} k^{-x} \le \int_3^\infty \frac{ds}{s^x} = \frac{1}{(x-1)3^{x-1}},$$

hence $\sum\limits_{k=4}^{\infty} k^{-x} = O\big(3^{-x}\big)$ and $\zeta(x) = 1 + 2^{-x} + O\big(3^{-x}\big)$. Moreover,

$$\begin{aligned} \zeta(x)^{\zeta(x)} &= \exp\Big(\Big(1 + 2^{-x} + O\big(3^{-x}\big)\Big) \cdot \ln\Big(1 + 2^{-x} + O\big(3^{-x}\big)\Big)\Big) \\ &= \exp\Big(\Big(1 + 2^{-x} + O\big(3^{-x}\big)\Big) \cdot \Big(2^{-x} + O\big(3^{-x}\big)\Big)\Big) \\ &= \exp\Big(2^{-x} + O\big(3^{-x}\big)\Big) = 1 + 2^{-x} + O\big(3^{-x}\big). \end{aligned}$$

Finally, we have

$$\begin{aligned} &\lim_{x\to\infty} 2^x \Big(\zeta(x+k)^{\zeta(x+k)} - \zeta(x+l)^{\zeta(x+l)}\Big) \\ = &\lim_{x\to\infty} 2^x \Big(2^{-x-k} - 2^{-x-l} + O\big(3^{-x}\big)\Big) = 2^{-k} - 2^{-l}. \end{aligned}$$

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

3388. [2008 : 432, 434] *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA, in memory of Murray S. Klamkin.*

For all real $x \ge 1$, show that

$$\frac{1}{2}\sqrt{x-1} + \frac{(x-1)^2}{\sqrt{x-1} + \sqrt{x+1}} < \frac{x^2}{\sqrt{x} + \sqrt{x+2}}.$$

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, modified by the editor.

The inequality holds for $x = 1$, so let $x > 1$. Since $\sqrt{x+1} > \sqrt{x-1}$ and $\sqrt{x} + \sqrt{x+2} < 2\sqrt{x+1}$ by the concavity of $\sqrt{x}$, it suffices to prove

$$\frac{1}{2}\sqrt{x-1} + \frac{1}{2}(x-1)^{3/2} < \frac{x^2}{2\sqrt{x+1}}, \tag{1}$$

as the right side of (1) is less than the right side of the desired inequality and the left side of (1) is greater than the left side of the desired inequality. Inequality (1) is successively equivalent to

$$\begin{aligned} \sqrt{x^2-1} + \sqrt{x^2-1}(x-1) &< x^2, \\ x\sqrt{x^2-1} &< x^2, \\ \sqrt{x^2-1} &< x, \end{aligned}$$

and the last inequality is true. The proof is complete.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy (two solutions); ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incorrect solution was submitted.

STAN WAGON, Macalester College, St. Paul, MN, USA verified the inequality using a computer algorithm.

In this small space that remains, we again put out the call for more problem proposals from our readers in the areas of Geometry, Algebra, Logic, and Combinatorics. We have a vast store of interesting inequalities and we will continue to process them and accept new ones, but the other areas are wanting!

Regarding articles, we now have a small backlog. This is due to the diligence of the Articles Editor, James Currie, but also due to the fact that space for articles in ***Crux*** is extremely limited. For this reason we ask that, if possible, authors with articles appearing in ***Crux*** wait about 18–20 months before submitting another manuscript to ***Crux***. In the meantime, we will gear up and attempt to clear the backlog in 2010 by running the occasional 96 page issue.

Václav Linek