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ROMANIAN MATHEMATICAL COMPETITIONS
2008

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ROMANIAN MATHEMATICAL COMPETITIONS

2008

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FOREWORD

The 15^{th} volume of the Romanian Mathematical Contests series contains more than 200 problems given at different stages of the Romanian Mathematical Olympiad and other Romanian Contests. Most of them are original, but some problems from other sources were used as well during competitions.

Most of the problems were discussed by the contributors for a long time providing thus some significant comments in the text.

Some of the solutions belong to students and were given during contests; we thank them all.

We thank the Ministry of Education and Research for permanent involvement in supporting the Olympiads and the participation of our teams in international events.

Special thanks are due to SOFTWIN, Volvo Romania, Medicover, and WBS – sponsors of the Romanian IMO team. Thanks are also due to the "Sigma Foundation" for constant support in the mathematical competitions.

Luminiţa Stafi from "The Theta Foundation" has done the important job of carefully editing the text in this form.

Last, not least, we are grateful to the Board of the Institute of Mathematics "Simion Stoilow" in Bucharest, for constant technical support in the Mathematical Olympiads and involvement in the training seminars for students.

Bucharest, July 1st, 2008

Radu Gologan, Dan Schwarz

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PART ONE

PROBLEMS

THE 59th ROMANIAN MATHEMATICAL OLYMPIAD DISTRICT ROUND

March 5th, 2008

7th GRADE

Problem 1. Show that

$$n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) \geqslant (n+1)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right),$$

for all natural numbers $n \geqslant 1$.

Lucian Dragomir

Problem 2. Consider the square $ABCD$ and the point E on the side AB. The diagonal AC intersects the segment DE at point P. The perpendicular from point P to DE intersects side BC at point F. Prove that $EF = AE + FC$.

Virginia and Vasile Tică

Problem 3. In a school there are 10 classrooms. Each student in a classroom knows exactly one student in each of the other 9 classrooms. Prove that the number of students in each classroom is the same.

(Assume that if student A knows student B, then student B knows student A too.)

* * *

Problem 4. Let $M = \{1, 2, 4, 5, 7, 8, \ldots\}$ be the set of natural numbers not divisible by 3. The sum of $2n$ consecutive elements of set M is 300. Determine the possible values of n.

Ion Tiotioi

8th GRADE

Problem 1. If the intersection of a regular tetrahedron and a plane is a rhombus, prove that the rhombus must be a square.

* * *

Problem 2. Determine the irrational numbers x such that both $x^2 + 2x$ and $x^3 - 6x$ are rational numbers.

Virginia and Vasile Tică

Problem 3. Consider the cube $ABCDA'B'C'D'$ and the points M, N, P, such that M is the foot of the perpendicular from A to plane $(A'CD)$, N is the foot of the perpendicular from B to the diagonal $A'C$, and P is the symmetric of D with respect to C. Show that the points M, N, P are collinear.

Ion Tiotioi

Problem 4. Determine the strictly positive real numbers x, y, z that satisfy simultaneously the conditions: $x^3y + 3 \leqslant 4z$, $y^3z + 3 \leqslant 4x$, and $z^3x + 3 \leqslant 4y$.

Dan Nedeianu

9th GRADE

Problem 1. Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers such that $|a_{n+1} - a_n| \leqslant 1$, for all $n \in \mathbb{N}^*$, and $(b_n)_{n\geqslant 1}$ the sequence defined by

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Show that $|b_{n+1} - b_n| \leqslant \frac{1}{2}$, for all $n \in \mathbb{N}^*$.

Dan Marinescu and Aurel Cornea

Problem 2. Consider the set $A = \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$, $n \geqslant 6$. Show that A is the union of three pairwise disjoint sets, with the same cardinality and the same sum of their elements, if and only if n is a multiple of 3.

* * *

Problem 3. Show that if $n \geqslant 4$, $n \in \mathbb{N}$ and $\left[\frac{2^n}{n}\right]$ is a power of 2, then n is a power of 2.

Radu Gologan

Problem 4. Let $ABCD$ be a quadrilateral that can be inscribed in a circle. Denote by P the intersection point of lines AD and BC, and by Q the intersection point of lines AB and CD. Let E be the fourth vertex of the parallelogram $ABCE$, and F the intersection of lines CE şi PQ. Prove that the points D, E, F, and Q lie on the same circle.

United Kingdom, shortlist for Romanian Master in Mathematics 2008

10th GRADE

Problem 1. Let a and b be two complex numbers. Prove the inequality

$$|1 + ab| + |a + b| \geqslant \sqrt{|a^2 - 1| \cdot |b^2 - 1|}.$$

Dan Nedeianu

Problem 2. Determine the integers x such that

$$\log_3(1 + 2^x) = \log_2(1 + x).$$

Lucian Dragomir

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{2}, \quad \text{for all } x, y \in \mathbb{R}.$$

a) Prove that the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = f(x) - f(0)$ is additive, i.e. $g(x + y) = g(x) + g(y)$, for all $x, y \in \mathbb{R}$.

b) Show that f is a constant function.

Dorel Miheţ

Problem 4. Let $n \geqslant 3$ be an integer and $z = \cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n}$. Consider the sets

$$A = \{1, z, z^2, \ldots, z^{n-1}\}$$

and

$$B = \{1, 1 + z, 1 + z + z^2, \ldots, 1 + z + \cdots + z^{n-1}\}.$$

Determine the set $A \cap B$.

Marcel Ţena

11^{th} GRADE

Problem 1. If $A \in \mathcal{M}_2(\mathbb{R})$, show that

$$\det(A^2 + A + I_2) \geqslant \frac{3}{4}(1 - \det A)^2.$$

Dan Nedeianu

Problem 2. Consider $A, B \in \mathcal{M}_n(\mathbb{R})$. Show that $\operatorname{rank} A + \operatorname{rank} B \leqslant n$ if and only if there exists an invertible matrix $X \in \mathcal{M}_n(\mathbb{R})$, such that $AXB = O_n$.

* * *

Problem 3. Let $(x_n)_{n\geqslant 1}$, $(y_n)_{n\geqslant 1}$ be two sequences of strictly positive numbers, such that

$$x_{n+1} \geqslant \frac{x_n + y_n}{2}, \quad y_{n+1} \geqslant \sqrt{\frac{x_n^2 + y_n^2}{2}}, \quad \text{for all } n \in \mathbb{N}^*.$$

a) Show that the limits of the sequences $(x_n + y_n)_{n\geqslant 1}$ and $(x_n y_n)_{n\geqslant 1}$ exist.

b) Show that the limits of the sequences $(x_n)_{n\geqslant 1}$, $(y_n)_{n\geqslant 1}$ exist and are equal.

Dan Marinescu

Problem 4. Determine for what values of $a \in [0, \infty)$ there exist continuous functions $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(f(x)) = (x - a)^2, \quad \text{for all } x \in \mathbb{R}.$$

Dorel Miheţ

12^{th} GRADE

Problem 1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)\mathrm{d}x = \int_0^1 xf(x)\mathrm{d}x$. Show that there exists $c \in (0,1)$ such that

$$f(c) = \int_0^c f(x)\mathrm{d}x.$$

Cezar Lupu

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, periodic function of period T. If F is an antiderivative of f, show that:

a) the function $G : \mathbb{R} \to \mathbb{R}$ given by

$$G(x) = F(x) - \frac{x}{T}\int_0^T f(t)\mathrm{d}t$$

is periodic;

b)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{F(k)}{n^2 + k^2} = \frac{\ln\sqrt{2}}{T}\int_0^T f(x)\mathrm{d}x.$$

Dan Nedeianu

Problem 3. Let A be a commutative unitary ring with an odd number of elements. If n is the number of solutions to the equation $x^2 = x$, $x \in A$, and m is the number of invertible elements in A, show that n divides m.

Mihai Piticari

Problem 4. Let K be a finite field. We say that two polynomials f and g in $K[X]$ are ***neighbours*** if they have the same degree and they differ by exactly one coefficient.

a) Show that all neighbours of the polynomial $X^2 + \hat{1} \in \mathbb{Z}_3[X]$ can be factored out in $\mathbb{Z}_3[X]$.

b) If the number of elements in K is $q \geqslant 4$, show that any polynomial of degree $q - 1$ in $K[X]$ has a neighbour that can be factored out in $K[X]$ and also has a neighbour with no roots in K.

Marian Andronache

THE 59th ROMANIAN MATHEMATICAL OLYMPIAD FINAL ROUND

Timişoara, April 30th, 2008

7th GRADE

Problem 1. The acute triangle ABC has $B > C$. Consider the altitude AD, $D \in BC$, and the perpendicular DE to AC, $E \in AC$. Consider the point F on the segment DE. Prove that the lines AF and BF are perpendicular if and only if $EF \cdot DC = BD \cdot DE$.

Vasile Pop

Problem 2. Given that a rectangle can be divided into 200 and into 288 equal squares, prove that it can also be divided into 392 equal squares.

Marius Perianu

Problem 3. Let p, q and r be three prime numbers such that $5 \leqslant p < q < r$. Given that $2p^2 - r^2 \geqslant 49$ and $2q^2 - r^2 \leqslant 193$, find p, q, r.

Mircea Fianu

Problem 4. Let $ABCD$ be a rectangle of center O. Assume that $AB \neq BC$. The perpendicular line at O on BD intersects the lines AB and BC at points E and F, respectively. Let M and N be the midpoints of the segments CD and AD, respectively. Prove that $FM \perp EN$.

Dinu Şerbănescu

8th GRADE

Problem 1. The lengths of the edges of a tetrahedron are natural numbers, such that the product of lengths of any pair of opposite edges is equal to 6. Show

that the tetrahedron is a regular triangular pyramid with the property that the angle between a lateral edge and the plane of the base is larger than or equal to $30°$.

Manuela Prajea

Problem 2. A sequence of four even decimal digits, no digit of which occurs three or four times, is called *admissible*.

a) Determine the number of admissible sequences.

b) For every natural number n, $n \geqslant 2$, we denote by d_n the number of ways to complete a table with n rows and 4 columns whose entries are even decimal digits, such that the following conditions are fulfilled:

i) every row is an admissible sequence;

ii) the admissible sequence 2, 0, 0, 8 occurs on a single row of the table.

Determine the values of n such that the number $\frac{d_{n+1}}{d_n}$ is an integer.

Nicolae Stăniloiu

Problem 3. Let $a, b \in [0, 1]$. Prove the inequality:

$$\frac{1}{1+a+b} \leqslant 1 - \frac{a+b}{2} + \frac{ab}{3}.$$

Lucian Dragomir

Problem 4. Consider the cube $ABCDA'B'C'D'$. On the edges $(A'D')$, $(A'B')$, and $(A'A)$ consider the points M_1, N_1, and P_1, respectively. On the edges (CB), (CD), and (CC') consider the points M_2, N_2, and P_2, respectively. Denote by d_1 the distance between the lines M_1N_1 and M_2N_2, by d_2 the distance between the lines N_1P_1 and N_2P_2, and by d_3 the distance between the lines P_1M_1 and P_2M_2. Suppose that the distances d_1, d_2, and d_3 are pairwise distinct. Show that the lines M_1M_2, N_1N_2, and P_1P_2 are concurrent.

Mircea Fianu

9th GRADE

Problem 1. Determine the functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(x^2 + f(y)) = xf(x) + y$, for all $x, y \in \mathbb{N}$.

Lucian Dragomir

Problem 2. a) Show that $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{2n}}>n, \forall n\in\mathbb{N}^*$.
b) Prove that

$$\min\left\{k\in\mathbb{N},\ k\geqslant 2;\ \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}>n\right\}>2^n,$$

for all $n\in\mathbb{N}^*$.

Dan Marinescu and Vasile Cornea

Problem 3. Consider $n\in\mathbb{N}^*$ and the real numbers a_i, $i=1,2,\ldots,n$ with $|a_i|\leqslant 1$ and $\sum\limits_{i=1}^{n}a_i=0$.

Show that $\sum\limits_{i=1}^{n}|x-a_i|\leqslant n$, for all $x\in\mathbb{R}$ such that $|x|\leqslant 1$.

Radu Gologan

Problem 4. On the sides of triangle ABC consider the points $C_1,C_2\in(AB)$, $B_1,B_2\in(AC)$, $A_1,A_2\in(BC)$ such that triangles $A_1B_1C_1$ and $A_2B_2C_2$ have the same centroid.

Show that the sets $[A_1B_1]\cap[A_2B_2]$, $[B_1C_1]\cap[B_2C_2]$, $[C_1A_1]\cap[C_2A_2]$ are nonempty.

Dinu Şerbănescu

10th GRADE

Problem 1. Consider the triangle ABC and the points $D\in(BC)$, $E\in(CA)$, $F\in(AB)$, such that

$$\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}.$$

Prove that if the circumcenters of triangles DEF and ABC coincide, then the triangle ABC is equilateral.

Dana Heuberger

Problem 2. Let a,b,c be three complex numbers such that $a|bc|+b|ca|+c|ab|=0$. Prove that

$$|(a-b)(b-c)(c-a)|\geqslant 3\sqrt{3}|abc|.$$

Bogdan Enescu

Problem 3. Consider the set $A=\{1,2,3,\ldots,2008\}$. We say that a set is *of type* r, $r\in\{0,1,2\}$, if that set is a nonempty subset of A, and the sum of its elements yields the remainder r when divided by 3. Denote by X_r, $r\in\{0,1,2\}$ the class of sets of type r.

Determine which of the classes X_r, $r\in\{0,1,2\}$, is the largest.

Mihai Bălună and Vasile Pop

Problem 4. Consider the statement $p(n):(n^2+1)|n!$, $n\in\mathbb{N}$. Show that the sets

$$A=\{n\in\mathbb{N}\mid p(n)\text{ is true}\}\quad\text{and}\quad F=\{n\in\mathbb{N}\mid p(n)\text{ is false}\}$$

are infinite.

Gheorghe Iurea

11th GRADE

Problem 1. Let $f:(0,\infty)\to\mathbb{R}$ be a continuous function, such that for any $x\in(0,\infty)$ the sequence $(f(nx))_{n\in\mathbb{N}^*}$ is nondecreasing.

Prove that f is a nondecreasing function.

Radu Gologan

Problem 2. Prove that an invertible matrix $A\in\mathcal{M}_n(\mathbb{C})$ has the property $A^{-1}=\overline{A}$ if and only if there exists an invertible matrix $B\in\mathcal{M}_n(\mathbb{C})$ such that $A=B^{-1}\cdot\overline{B}$.

Vasile Pop

Problem 3. Let $f:\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on $\mathbb{R}$ such that there exists $c\in\mathbb{R}$ with

$$\frac{f(b)-f(a)}{b-a}\neq f'(c),\quad\text{for all } a,b\in\mathbb{R},\ a\neq b.$$

Prove that $f''(c)=0$.

Bogdan Enescu

Problem 4. Let $A\in\mathcal{M}_n(\mathbb{R})$ be an antisymmetric matrix ($\forall i,j,\ a_{ij}+a_{ji}=0$). Prove that

$$\det(A+xI_n)\cdot\det(A+yI_n)\geqslant\det(A+\sqrt{xy}I_n)^2,$$

for all $x, y \in [0, \infty)$.

Octavian Ganea

12th GRADE

Problem 1. Let a be a positive real number and let $f : [0, \infty) \to [0, a]$ be a function which has the intermediate value property on $[0, \infty)$ and is continuous on $(0, \infty)$. If $f(0) = 0$ and

$$xf(x) \geqslant \int_0^x f(t)\mathrm{d}t, \quad \text{for all } x \in (0, \infty),$$

prove that f has antiderivatives on $[0, \infty)$.

Dorin Andrica

Problem 2. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function, whose derivative f' is continuous on $[0, 1]$. Prove that if $f(1/2) = 0$, then

$$\int_0^1 (f'(x))^2 \,\mathrm{d}x \geqslant 12 \left(\int_0^1 f(x)\mathrm{d}x \right)^2.$$

Cezar Lupu

Problem 3. Let A be a finite unitary ring with n elements, such that the equation $x^n = 1$ has the unique solution $x = 1$ in A. Prove that:

(a) 0 is the unique nilpotent element of the ring A;

(b) there exists $k \in \mathbb{N}$, $k \geqslant 2$, such that the equation $x^k = x$ has n solutions in A.

($x \in A$ is *nilpotent* if there exists $m \in \mathbb{N}^*$ such that $x^m = 0$.)

Dan Schwarz

Problem 4. Let $\mathcal{G}$ be the set of finite groups with at least two elements.

(a) Show that if $G \in \mathcal{G}$, then

$$|\mathrm{End}(G)| \leqslant \sqrt[p]{n^n},$$

where $|\mathrm{End}(G)|$ is the number of endomorphisms of G, $n = n(G)$ is the number of elements of G, and $p = p(G)$ is the greatest prime divisor n.

(b) Determine the groups in $\mathcal{G}$ such that the inequality in (a) holds with equality.

Marian Andronache

SELECTION TESTS FOR THE BALKAN AND INTERNATIONAL MATHEMATICAL OLYMPIADS

FIRST SELECTION TEST

Problem 1. Determine all families $\mathcal{F}$ of $n \geqslant 1$ integers such that no sum of elements of a non-empty subfamily of $\mathcal{F}$ is divisible by $n + 1$.

(How many such families exist, made of distinct positive integers between 1 and $n^2 + n$?)

Dan Schwarz

Problem 2. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers so that $a_i < b_i$, for all $i = 1, 2, \ldots, n$, and $b_1 + b_2 + \cdots + b_n < 1 + a_1 + a_2 + \cdots + a_n$. Prove there exists $c \in \mathbb{R}$ such that

$$(a_i + k + c)(b_i + k + c) > 0,$$

for all $i = 1, 2, \ldots, n$ and $k \in \mathbb{Z}$.

Vasile Pop

Problem 3. A convex hexagon $ABCDEF$ has all sides of length 1. Prove that one of the radii of the circumscribed circles of the triangles ACE and BDF is at least 1 long.

Kvant

Problem 4. Prove that, given any convex polygon P with n sides, there exists a set S of $n-2$ points interior to P, such that the interior of any triangle determined by three of the vertices of P contains exactly one point from S.

American Mathematical Monthly

Problem 5. Determine the greatest common divisor of the numbers

$$2^{561}-2, 3^{561}-3, \dots, 561^{561}-561.$$

Dorin Andrica and Mihai Piticari

SECOND SELECTION TEST

Problem 6. Let $n \geqslant 3$ be an odd integer. Determine the maximum value of the cyclic sum, for $0 \leqslant x_i \leqslant 1$, $i = 1, 2, \dots, n$,

$$E = \sqrt{|x_1 - x_2|} + \sqrt{|x_2 - x_3|} + \dots + \sqrt{|x_{n-1} - x_n|} + \sqrt{|x_n - x_1|}.$$

American Mathematical Monthly

Problem 7. Does it exist a sequence of integers $1 \leqslant a_1 < a_2 < a_3 < \cdots$ such that, for any integer n, the set $\{a_k + n; k = 1, 2, 3, \dots\}$ contains a finite number of primes?

American Mathematical Monthly

Problem 8. Prove that any convex pentagon has a vertex whose distance to the support line of its opposite side is strictly less than the sum of the distances from its neighbouring vertices to the same line.

American Mathematical Monthly

Problem 9. Determine the minimum number of edges that a connected graph with $n \geqslant 3$ vertices may have, if each edge belongs to at least one triangle.

American Mathematical Monthly

THIRD SELECTION TEST

Problem 10. Let triangle ABC have $BC < AB$, and let points D on (AC), E on (AB) be such that $\angle DEB = \angle DCB$. It is given that point F lies in the interior of the quadrilateral $BCDE$, and the pairs of circumcircles of triangles BEF, CDF, respectively BCF, DEF, are tangent. Prove that points A, C, E, F, are concyclic.

Adapted after Cosmin Pohoaţă

Problem 11. Let ABC be an acute-angled triangle, H its orthocenter and X any point in the plane. The circle of diameter HX meets the second time the line AH at point A_1, and the line AX at point A_2. Points B_1, B_2 and C_1, C_2 are defined in a similar way. Prove that the lines A_1A_2, B_1B_2, C_1C_2 are concurrent.

Kvant

Problem 12. For m and n odd integers larger than 1, prove that $2^m - 1$ does not divide $3^n - 1$.

American Mathematical Monthly

Problem 13. A group of people is said to be *n-balanced* if in any subgroup of 3 people there exists (at least) a pair acquainted with each other, and if in any subgroup of n people there exists (at least) a pair not acquainted with each other.

i) Prove that the number of people in a n-balanced group has an upper bound. We may then denote by p_n the maximal possible number of people in a n-balanced group.

ii) Prove that $p_n \leqslant \frac{(n-1)(n+2)}{2}$.

iii) Compute, with proof, p_3, p_4 and p_5.

iv) Prove that $p_6 \leqslant 18$.

Adapted after Dorel Miheţ

FOURTH SELECTION TEST

Problem 14. Consider the convex quadrilateral $ABCD$ with non-parallel opposite sides. Let O be the meeting point of lines AC and BD, P be the meeting point of lines AB and CD, and Q be the meeting point of lines AD and BC. Let R be the foot of the perpendicular from O onto PQ, and M, N, S, respectively T, the feet of the perpendiculars from R onto CD, BC, DA, respectively AB.

Prove that the points M, N, S and T are concyclic.

Dan Barbilian

Problem 15. Given co-prime positive integers m, n, and integer s, compute the number of subsets $\{x_1, x_2, \dots, x_m\} \subseteq \{1, 2, \dots, m+n-1\}$ having

$$x_1 + x_2 + \dots + x_m \equiv s \pmod{n}.$$

American Mathematical Monthly

Problem 16. For positive integer $n \geqslant 2$, prove that in any selection of at least $2^{n-1}+1$ non-empty distinct subsets of $\{1,2,\ldots,n\}$ there are three such that one of them is the union of the two other.

M. Cavachi

FIFTH SELECTION TEST

Problem 17. For what positive integers n does there exist a permutation σ of $\{1,2,\ldots,n\}$ such that the differences $|\sigma(k)-k|$, $1 \leqslant k \leqslant n$, are all distinct?

American Mathematical Monthly

Problem 18. Let ABC be a triangle, and $\mathcal{K}_a$, $\mathcal{K}_b$, $\mathcal{K}_c$ be the circles having its medians as diameters. Show that if two of these circles are tangent to the incircle of the triangle, then the third one is also tangent to the incircle.

Dinu Şerbănescu

Problem 19. Let $f(n)$ denote the maximum number of disjoint rectangles that the unit square can be partitioned into, such that any horizontal or vertical line intersects the interior of at most n rectangles. Show that

$$3 \cdot 2^{n-1} - 2 \leqslant f(n) \leqslant 3^n - 2.$$

(It is assumed that all the rectangles have sides parallel to the sides of the given square.)

American Mathematical Monthly

SELECTION TESTS FOR THE JUNIOR BALKAN MATHEMATICAL OLYMPIAD

FIRST SELECTION TEST

Problem 1. Let p be a prime number, $p \neq 3$, and let a, b be integer numbers so that $p \mid a+b$ and $p^2 \mid a^3+b^3$. Show that $p^2 \mid a+b$ or $p^3 \mid a^3+b^3$.

* * *

Problem 2. Prove that for any positive integer n there exists a multiple of n whose decimal digits add up to n.

Mihai Bălună

Problem 3. Let ABC be an acute-angled triangle. Consider the equilateral triangle $A'UV$, with $A' \in (BC)$, $U \in (AC)$, $V \in (AB)$ such that $UV \parallel BC$. The points $B' \in (AC)$ and $C' \in (AB)$ are defined similarly. Show that the lines AA', BB' and CC' are concurrent.

Vasile Pop

Problem 4. Let ABC be a triangle and D the midpoint of BC. On the sides AB and AC there are points M, N respectively, other than the midpoints of these segments, so that $AM^2+AN^2=BM^2+CN^2$ and $\angle MDN = \angle BAC$. Prove that $A=90^\circ$.

Francisc Bozgan

Problem 5. Let $n \in \mathbb{N}$, $n \geqslant 2$ and let $a_1, a_2, \ldots a_n$ be integer numbers such that $0 < a_k \leqslant k$, for all $k=1,2,\ldots,n$. If $a_1+a_2+\cdots+a_n$ is even, prove that

$$a_1 \pm a_2 \pm \ldots \pm a_n = 0,$$

for some choice of the signs "+" and "-".

SECOND SELECTION TEST

Problem 6. Consider an acute-angled triangle ABC, the height AD and the point E where the diameter through A of the circumcircle meets the line BC. Let M, N be the reflected images of D across the lines AC and AB. Show that $\angle EMC = \angle BNE$.

Dinu Şerbănescu

Problem 7. Let $a_1, a_2, \ldots, a_n$ be a sequence of integers such that a_k is the number of multiples of k in the sequence, for all $k = 1, 2, \ldots, n$. Find all possible values of n.

Cristian Mangra

Problem 8. Let $n \in \mathbb{N}^*$ and let $a_1, a_2, \ldots, a_n$ be positive real numbers so that

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}.$$

Prove that for any $m = 1, 2 \ldots, n$, there exist m numbers among a_i whose sum is at least m.

Andrei Ciupan and Flavian Georgescu

Problem 9. Let a, b be real numbers with the property that the integer part of $an + b$ is an even number, for all $n \in \mathbb{N}$. Show that a is an even integer.

Dinu Şerbănescu

THIRD SELECTION TEST

Problem 10. Ten numbers are chosen at random from the set $1, 2, 3, \ldots, 37$. Show that one can select four distinct numbers from the chosen ones so that the sum of two of them is equal to the sum of the other two.

Vasile Pop

Problem 11. Let a, b, c be positive real numbers with $ab + bc + ca = 3$. Prove that

$$\frac{1}{1 + a^2(b+c)} + \frac{1}{1 + b^2(c+a)} + \frac{1}{1 + c^2(a+b)} \leqslant \frac{1}{abc}.$$

Vlad Matei

Problem 12. Find all primes p, q satisfying the equation $2p^q - q^p = 7$.

Francisc Bozgan

Problem 13. Let d be a line and let M, N be two points on d. Circles $\alpha, \beta, \gamma, \delta$ centered at A, B, C, D are tangent to d in such a manner that circles α, β are externally tangent at M, while circles γ, δ are externally tangent at N. Moreover, points A and C lies on the same side of line d. Prove that if there exists a circle tangent to all circles $\alpha, \beta, \gamma, \delta$, containing all of them in the interior, then lines AC, BD and d are concurrent or parallel.

Flavian Georgescu

FOURTH SELECTION TEST

Problem 14. Let $ABCD$ be a quadrilateral with no two opposite sides parallel. The parallel from A to BD meets the line CD at point F and the parallel from D at AC meet the line AB at point E. Consider the midpoints M, N, P, Q of the segments AC, BD, AF, DE respectively. Show that lines MN, PQ and AD are concurrent.

Dinu Şerbănescu

Problem 15. Let $m, n \in \mathbb{N}^*$ and $A = \{1, 2, \ldots, n\}$, $B = \{1, 2, \ldots, m\}$. A subset S of the set product $A \times B$ has the property that for any pairs $(a, b), (x, y) \in S$, then $(a - x)(b - y) \leqslant 0$. Show that S has at most $m + n - 1$ elements.

Dinu Şerbănescu

Problem 16. Find all pairs of integers $(m, n), n, m > 1$ so that $mn - 1$ divides $n^3 - 1$.

Francisc Bozgan

Problem 17. Determine the maximum value of the real number k such that

$$(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} - k\right) \geqslant k,$$

for all real numbers $a, b, c \geqslant 0$, not all zero, with $a + b + c = ab + bc + ca$.

Andrei Ciupan

THE 25$^{\text{th}}$ BALKAN MATHEMATICAL OLYMPIAD

Ohrid, Macedonia, May 5 – May 10, 2008

Problem 1. Let ABC be a scalene acute-angled triangle with $AC > BC$. Let O be its circumcenter, H its orthocenter and F the foot of the altitude from C. Let P be the point (other than A) on the line AB for which $AF = PF$, and M the midpoint of the side AC. PH and BC meet at X, OM and FX meet at Y, and OF and AC meet at Z. Prove that points F, M, Y and Z are concyclic.

Cyprus

Problem 2. Does it exist a sequence $a_1, a_2, \ldots, a_n, \ldots$ of positive real numbers, which simultaneously satisfies

(i) $\sum\limits_{i=1}^{n} a_i \leqslant n^2$, for all positive integers n;

(ii) $\sum\limits_{i=1}^{n} \frac{1}{a_i} \leqslant 2008$, for all positive integers n?

Bulgaria

Problem 3. Let n be a positive integer. Rectangle $ABCD$, having the lengths of its sides $AB = 90n + 1$ and $BC = 90n + 5$, is partitioned in unit squares with sides parallel with the sides of the rectangle. Let S be the set of all points which are vertices of these unit squares. Prove that the number of distinct lines passing through at least two points of S is divisible by 4.

Bulgaria

Problem 4. Let c be a positive integer. The sequence $(a_n)_{n\geqslant 1}$ is defined by $a_1 = c$, $a_{n+1} = a_n^2 + a_n + c^3$, for all positive integers n. Determine all values of c for which there exist integers $k \geqslant 1$, $m \geqslant 2$ such that $a_k^2 + c^3$ be the power m of some integer.

Bulgaria

THE FIRST "ROMANIAN MASTER IN MATHEMATICS" COMPETITION

Bucharest, T. Vianu Highschool, February 2008

Problem 1. Let ABC be an equilateral triangle. P is a variable point internal to the triangle and its perpendicular distances onto the sides are denoted by a^2, b^2 and c^2 for positive real numbers a, b and c. Find the locus of points P so that a, b and c can be the sides of a non-degenerate triangle.

United Kingdom

Problem 2. Given positive integer $a > 1$, prove that any positive integer N has a multiple in the sequence

$$(a_n)_{n\geqslant 1}, \quad a_n = \left\lfloor \frac{a^n}{n} \right\rfloor.$$

Romania – Dan Schwarz

Problem 3. Prove that any one-to-one surjective function $f : \mathbb{Z} \to \mathbb{Z}$ can be written as $f = u + v$ where $u, v : \mathbb{Z} \to \mathbb{Z}$ are one-to-one surjective functions.

Romania – Ion Savu and Sorin Rădulescu

Problem 4. Prove that from among any $(n+1)^2$ points inside a square of side-length positive integer n, one can pick three, such that the triangle determined by them has area no more than $\frac{1}{2}$.

Romania – Dan Schwarz

THE FIFTH "IMAR" MATHEMATICAL COMPETITION

Institute of Mathematics "Simion Stoilow", Bucharest, October 2007

Problem 1. For real numbers $x_i > 1$, $1 \leqslant i \leqslant n$, $n \geqslant 2$, such that

$$\frac{x_i^2}{x_i - 1} \geqslant S = \sum_{j=1}^{n} x_j, \quad \text{for all } i = 1, 2, \ldots, n$$

find, with proof, sup S.

Adapted after Moldova Olympiad

Problem 2. Denote by $\mathcal{C}$ the family of all configurations C of $N > 1$ distinct points on the sphere S^2, and by $\mathcal{H}$ the family of all closed hemispheres H of S^2. Compute

$$\max_{H \in \mathcal{H}} \min_{C \in \mathcal{C}} |H \cap C|, \qquad \min_{H \in \mathcal{H}} \max_{C \in \mathcal{C}} |H \cap C|,$$

$$\max_{C \in \mathcal{C}} \min_{H \in \mathcal{H}} |H \cap C| \quad \text{and} \quad \min_{C \in \mathcal{C}} \max_{H \in \mathcal{H}} |H \cap C|.$$

Dan. Schwarz

Problem 3. Prove that among $N \geqslant 2n - 2$ integers, of absolute value not higher than $n > 2$, and of absolute value of their sum S less than $n - 1$, there exist some of sum zero.

Show that for $|S| = n - 1$ this is not anymore true, and neither for $N = 2n - 3$ (when even for $|S| = 1$ this is not anymore true).

Adapted after Canada Olympiad

THE "MATH STARS" MATHEMATICAL COMPETITION

Bucharest, ICHB Highschool, December 2007

FIRST DAY

Problem 1. Show that for any positive integer n there exists a positive integer m such that

$$(1 + \sqrt{2})^n = \sqrt{m} + \sqrt{m + 1}.$$

1989 IMO Long List

Problem 2. Determine the positive integers n, x and y for which

$$2^x - n^{y+1} = \pm 1.$$

Dan Schwarz

Problem 3. Let ABC be a triangle and A_1, B_1, C_1 be the feet of the altitudes from A, B, C. Let A_2, respectively A_3, be the orthogonal projection of A_1 onto AB, respectively AC; points B_2, B_3 and C_2, C_3 are defined in an analogous way. The lines B_2B_3 and C_2C_3 meet at A_4, the lines C_2C_3 and A_2A_3 meet at B_4, while the lines A_2A_3 and B_2B_3 meet at C_4. Show that the lines AA_4, BB_4 and CC_4 are concurrent.

Lucian Ţurea

Problem 4. At a table-tennis tournament, the $n \geqslant 2$ participants play, each against each, exactly one match. Show that exactly one of the following two situations occurs at the end of the tournament:

(1) the n participants can be labeled with the numbers $1, 2, \ldots, n$ such that 1 beat 2, 2 beat 3, and so on, $n - 1$ beat n and n beat 1;

(2) the n participants can be partitioned in two non-empty sets A, B, such that every member of A beat each member of B.

1989 IMO Long List

SECOND DAY

Problem 5. Show there uniquely exists a function $f : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ which simultaneously satisfies the following three conditions:

(1) $f(x, y) = f(y, x)$, for all $x, y \in \mathbb{N}^*$;
(2) $f(x, x) = x$, for all $x \in \mathbb{N}^*$; and
(3) $(y - x)f(x, y) = yf(x, y - x)$, for all $x, y \in \mathbb{N}^*$, $y > x$.

* * *

Problem 6. Let have $n > 3$ points in the space, four by four non-coplanar, any two of them connected by wires.

(1) By cutting the $n - 1$ wires that connect one point from the others, that point is disconnected (becomes *isolated*). Show that cutting less than $n - 1$ wires does not disconnect the structure.

(2) Determine the minimum number of wires needed to be cut, in order to disconnect the structure, with no point becoming isolated.

Adapted after Virginia Tech Contest

Problem 7. Let $A_0 \cdots A_{n-1}$ be a regular n-gon. For each index i, consider a point B_i lying on the side A_iA_{i+1}, such that $A_iB_i < \frac{1}{2}A_iA_{i+1}$, and a point C_i lying on the segment A_iB_i (indices are reduced modulo n). Show that the perimeter of the polygon $C_0 \cdots C_{n-1}$ is at least as large as the perimeter of the polygon $B_0 \cdots B_{n-1}$.

American Mathematical Monthly

Problem 8. Prove that any set of 27 positive integers, ranging between 1 and 2007, contains three distinct elements a, b, c such that $\gcd(a, b)$ (the greatest common divisor of a and b) divides c.

Open Question. Improve this result, by lowering the number 27 necessary to obtain the stated property.

Dan Schwarz

THE "CLOCK-TOWER SCHOOL" SENIORS COMPETITION

Rm. Vâlcea, January 2008

FIRST DAY

Problem 1. Prove that, for any $n \in \mathbb{N}$, $n \geqslant 2$, the Diophantine equation

$$1 + x_1^2 + \cdots + x_n^2 = y^2$$

has infinitely many positive integer solutions with $1 < x_1 < \cdots < x_n$.

Adapted after Dorel Miheţ

Problem 2. Let ABC be an acute-angled triangle, and ω, respectively Ω, be its incircle and circumcircle. Circle ω_A is tangent (internal) to Ω at A, and tangent (external) to ω at A_1. Points B_1 and C_1 are similarly obtained, starting with B, respectively C. Prove that lines AA_1, BB_1 and CC_1 are concurrent.

* * *

Problem 3. In the Cartesian coordinate plane define the strips

$$S_n := \{(x, y);\ n \leqslant x < n + 1\},$$

for every integer n. Assume each strip is colored either white or black. Prove one can place any rectangle R, not a square, in the plane, such that its vertices share a same color.

Radu Gologan and Dan Schwarz

SECOND DAY

Problem 4. Let $(a_n)_{n \geqslant 0}$ be a real sequence having

$$a_{n+1} + a_{n-1} = |a_n|, \quad \text{for all } n \geqslant 1.$$

Prove the sequence is periodic.

Adapted after Vasile Pop

Problem 5. A rectangle D is partitioned in (more than one) rectangles having their sides parallel to those of rectangle D. It is given that any line parallel to one of the sides of D, and having common points with the interior of D, will also have common points with the interior of (at least) one of the rectangles in the partition. Prove that in this partition there is (at least) a rectangle that has **no** common points with the border of D.

2007 IMO Shortlist – Japan

Problem 6. Given an odd integer $n > 3$ not divisible by 3, show that there exist distinct odd, positive integers a, b, and c such that

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

American Mathematical Monthly

THE "CLOCK-TOWER SCHOOL" JUNIORS COMPETITION

Rm. Vâlcea, January 2008

FIRST DAY

Problem 1. Consider a circle of center O and a chord AB of it (not a diameter). Take a point T on the ray OB. The perpendicular at T onto OB meets the chord AB at C and the circle at D and E. Denote by S the orthogonal projection of T onto the chord AB. Show that $AS \cdot BC = TE \cdot TD$.

Problem 2. The last digit in the decimal representation of number $a^2 + ab + b^2$, with $a, b \in \mathbb{N}^*$, is 0. Find its second-to-last digit.

Problem 3. Partition a triangle into (smaller) triangles. Show that the sum of the lengths of the lesser altitudes of the triangles of the partition is at least equal to the length of the lesser altitude of the given triangle.

Problem 4. Consider any 25 points, three by three non-collinear, in the interior of a square of side length 3. Show that there exist four among them that form a quadrilateral perimeter less than 5.

SECOND DAY

Problem 5. A positive integer has, in its decimal representation, 2008 digits equal to 1, 2008 digits equal to 4, while the rest of its digits are equal to 0. Show that this number cannot be a perfect square.

Problem 6. Let $\mathcal{P}$ be the set of all points of the plane, and $O \in \mathcal{P}$ fixed. The function $f : \mathcal{P} \setminus \{O\} \to \mathbb{R}$ has the property:

for any four distinct points $A, B, C, D \in \mathcal{P} \setminus \{O\}$ *with* $\Delta AOB \sim \Delta COD$,

$$f(A) - f(B) + f(C) - f(D) = 0 \text{ occurs.}$$

Prove the function f is constant.

Problem 7. For any real numbers $a, b, c > 0$, with $abc = 8$, prove

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leqslant 0.$$

Problem 8. Let p be a prime, and q an integer, not divisible by p. Prove there exist infinitely many integers k such that pq divides $q^k + 1 - k$.

SHORTLISTED PROBLEMS FOR THE 2008 OLYMPIAD

JUNIORS

1. Consider a triangle ABC and the points $D \in (AB)$, $E \in (BC)$ and $F = AE \cap CD$. Given that AE is the bisector of $\angle BAC$, $2DB = AB$, $3EC = BC$, and $4FC = AB$, find the angles of the triangle ABC.

2. Let I_a be the common point of the external bisectors of the angles B and C of a triangle ABC. Denote D the orthogonal projection of I_a onto BC and $M = AI_a \cap BC$. Prove that:
a) $\frac{AB \cdot BC}{AC \cdot CD} = \frac{1+\cos C}{1+\cos B}$;
b) $(AB + BD)^2 \geqslant AM(2AI_a - AM)$.

3. A trapezoid $ABCD$ has $m(\angle DAB) = 90°$, its diagonals are perpendicular and meet at point O. The parallel through O at AB meets BC in P and the perpendicular from O onto BC intersects AD in R. Prove that $PR = AD$.

4. In triangle ABC, $(AD$ is the bisector of $\angle BAC$ and I_1, I_2 are the incenters of the triangles ABD and ADC, respectively. Prove that AD, BI_2 and CI_1 are concurrent.

5. Consider the trapezoid $ABCD$ $(AB \parallel CD,\ AB > CD)$, $DM \perp AB$, $M \in (AB)$ and let N be the midpoint of the diagonal (BD).
Prove that $MN \parallel AC$ if and only if the trapezoid is isosceles.

6. Given a parallelogram of area S, draw a parallelogram of area $\frac{S}{6}$, using only a straightedge.

7. Let $p \geqslant 2$ be a positive prime. Find all $n \in \mathbb{N}$ such that the number $\frac{n^{2008}}{n+p}$ is a prime integer.

8. Let $ABCD$ be a quadrilateral inscribed in the circle $\mathcal{C}$ of diameter (AD) and E a point of $\mathcal{C}$. Let M, N, P be the feet of the perpendiculars from E onto AB, BC and CD respectively. Prove that the triangle MNP is right-angled.

9. If $S = \{\frac{1}{abc} \mid \overline{abc} = x^2 + 3x + 2, x \in \mathbb{N}\}$, compute the sum of its elements.

10. Are there positive integer numbers a and b, such that both $\sqrt{a + 2\sqrt{b}}$ and $\sqrt{b + 2\sqrt{a}}$ be rational?

11. Is it possible to color the vertices of a 2009-gon red or blue, such that among any six consecutive vertices exactly there are blue or exactly one is red?

12. How many ten digit integers have the sum of their digits equal to their product?

13. Let $a_1, a_2, \ldots, a_{2001}$ be a permutation of $1, 2, \ldots, 2001$. Prove that the largest of the numbers ka_k, $1 \leqslant k \leqslant n$ is at least 1001^2.

14. The sum of $n \geqslant 5$ given reals equals 1.

a) Show that, no matter how we arrange them on a circle, the sum of the products of the n pairs of neighbours is at most $\frac{1}{4}$.

b) Show that it is possible to arrange the numbers on a circle such that the sum of the products of the n pairs of neighbours is at most $\frac{1}{5}$.

15. Consider a right parallelepiped $ABCDA'B'C'D'$ and M, N the centers of the faces $A'B'C'D'$ and $ADD'A'$, respectively. Prove that, if $AM \perp A'C$ and $C'N \perp BD'$, then the parallelepiped is a cube.

16. Let $VABCD$ be a regular pyramid of apex V, and let E, G, F, H be points on the lateral edges $(VA), (VB), (VC)$ and (VD) respectively, such that $EF \cap AC = \{P\}$ and $GH \cap BD = \{R\}$. The parallel through E to AC intersects VC in E_1 and the parallel through H to BD intersects VB in H_1. The parallel through G to AB meets VA in G_1 and the parallel through F to CD meets VD in F_1. Denote O the common point of AC and BD. If $HF_1 = EG_1$ and $\frac{FE_1}{PO} = \frac{GH_1}{RO}$, show that the points E, G, F, H are coplanar.

17. Find all $\overline{abcd}$, $a, c \neq 0$, such that $\frac{\sqrt{\overline{abcd}}}{\sqrt{\overline{ab}}+\sqrt{\overline{cd}}}$ is rational.

18. Prove that, if a, b, c are positive reals and $abc = 1$, then

$$\frac{a^2+b^2}{a^4+b^4} + \frac{b^2+c^2}{b^4+c^4} + \frac{c^2+a^2}{c^4+a^4} \leqslant a + b + c.$$

SENIORS

19. Show that, for every acute triangle, $\sum \frac{a}{\cos A} \geqslant 2(a + b + c)$.

20. Consider $n \in \mathbb{N}$, $n \geqslant 3$ and the set $A = \{1, 2, \ldots, n\}$. Find all positive integers k for which it is possible to find k distinct functions $f_i : A \to \{0, 1\}$, $1 \leqslant i \leqslant k$, such that the function $g : A \to \mathbb{R}$, $g = f_1 + f_2 + \cdots + f_k$ is injective.

21. The altitudes from the vertices A, B and C of a triangle meet again the circumcircle at points D, E and F, respectively. Prove that $\triangle ABC$ is equilateral in each of the following cases: i) the triangles BDC, CEA, AFB have the same perimeter; ii) the triangles BDC, CEA, AFB have the same area.

22. In the trapezoid $ABCD$ ($AB \parallel CD$), $M, N \in (CD)$ are the feet of the perpendiculars from A, respectively B onto CD. The circles $\mathcal{C}_1(r_1)$ and $\mathcal{C}_2(r_2)$, inscribed in the triangles AMD, respectively BNC, touch AM in P and BN in Q. Let t be the second common tangent of these circles. Show that $PQ \parallel t$ if and only if $AD = BC$ or $MN = 2\sqrt{r_1 r_2}$.

23. Let $ABCD$ be a quadrilateral and $k \in (0, 1)$. The points $M \in [BC]$, $N \in [AD]$, $P \in (MN)$ are variable and satisfy $\frac{CM}{BC} + \frac{DN}{AD} = 1$ and $\frac{MP}{MN} = k$. Find the locus of P.

24. Let $n \in \mathbb{N}$, $n \geqslant 2$ and $A = \{1, 2, \ldots, n\}$. Find the number of functions $f : A \to A$ satisfying $n \mid \sum\limits_{k=1}^{n} f(k)$.

25. Find all the reals x which can be written in the form

$$x = \frac{1}{a_1 a_2 \ldots a_n} + \frac{a_1}{a_2 a_3 \ldots a_n} + \frac{a_2}{a_3 a_4 \ldots a_n} + \cdots + \frac{a_{n-2}}{a_{n-1} a_n} + \frac{a_{n-1}}{a_n},$$

where $n > 0, a_1, a_2, \ldots, a_n$ are integers and $0 < a_1 < a_2 < \cdots < a_n$.

26. Let $ABCD$ be a convex quadrilateral and $A' \in (AB)$, $B' \in (BC)$, $C' \in (CD)$, $D' \in (DA)$ be such that $AA' = CC'$ and $BB' = DD'$. Show that the line through the midpoints of the segments (AC') and $(A'C)$ is perpendicular to the line through the midpoints of the segments (BD') and $(B'D)$ if and only if $ABCD$ is cyclic.

27. If $a, b, c > 0$ and $n \in \mathbb{N}$, show that

$$(a^{n+1}+b^{n+1}+c^{n+1})\left(\frac{1}{a^{n+1}}+\frac{1}{b^{n+1}}+\frac{1}{c^{n+1}}\right) \geqslant (a^n+b^n+c^n)\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right).$$

28. Show that, for every real numbers $x, y, z \geqslant 0$,

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geqslant \sqrt{4-\frac{14xyz}{(x+y)(y+z)(z+x)}}.$$

29. Show that if $a, b, c \geqslant 0$ and $a+b+c=1$, then

$$\frac{1}{3} \leqslant \frac{a}{a^2+a+1}+\frac{b}{b^2+b+1}+\frac{c}{c^2+c+1} \leqslant \frac{9}{13}.$$

30. Find all pairs (z, n), with $z \in \mathbb{C}$ and $n \in \mathbb{N}$, such that

$$z+z^2+\cdots+z^n = n|z|^n.$$

31. Let ABC be a right isosceles triangle and let M and N be points on the legs (AB) and (AC), respectively. Show that there exists a triangle with side lengths CM, BN, MN.

32. Find all the functions $f : (0,\infty) \to (0,\infty)$ satisfying

$$f(f(x)-x) = 6x, \quad \forall x \in (0,\infty).$$

33. Show that if $z \in \mathbb{C}$ and $|z| = 1$, then

$$\sum_{k=1}^{n}(n-k+1)|1+z^k| \geqslant \left\lfloor\frac{n}{2}\right\rfloor\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)|1-z|.$$

34. Let $a_1, a_2, \ldots, a_n \in (0,\infty)$ be such that $a_1+a_2+\cdots+a_n = 1$. Show that the function $f : [1,\infty) \to \mathbb{R}$, $f(x) = a_1^x + a_2^x + \cdots + a_n^x$ is strictly increasing.

35. Show that if $x_1, x_2, \ldots, x_n > 0$, then

$$\sqrt[n]{x_1x_2\ldots x_n} \leqslant \frac{\sqrt[n]{n^{n-1}x_1x_2\ldots x_n(x_1+x_2+\cdots+x_n)}}{\sqrt[n]{x_1}+\sqrt[n]{x_2}+\cdots+\sqrt[n]{x_n}} \leqslant \frac{x_1+x_2+\cdots+x_n}{n}.$$

36. Show that if $x, y, z \in (0,\infty)$, then

$$\frac{xy}{\sqrt{(x^2+z^2)(y^2+z^2)}}+\frac{yz}{\sqrt{(y^2+x^2)(z^2+x^2)}}+\frac{zx}{\sqrt{(z^2+y^2)(x^2+y^2)}} > 1.$$

37. An alphabet has a letters. Find the number of the words of length m having exactly p distinct letters.

38. Is it possible to find positive integers n and k such that $\lfloor(2+\sqrt{3})^{2n+1}\rfloor = \lfloor(4+\sqrt{15})^k\rfloor$?

39. Find all the positive integers a for which

$$M_a = \{f : \mathbb{N} \to \mathbb{N} \mid \underbrace{(f \circ f \circ \cdots \circ f)}_{k \text{ times } f}(n) = n+a, \quad \forall n \in \mathbb{N}\} \neq \emptyset.$$

40. Prove that, in every triangle,

$$9R^2 \geqslant a^2+b^2+c^2+(a-b)^2+(b-c)^2+(c-a)^2.$$

41. Let $a > 1$. Solve in $\mathbb{R}^*$ the equation $a^x + (2a+1)^{\frac{2}{x}} = (a+1)^2$.

42. Two regular n-gons $A_1A_2\ldots A_n$ and $B_1B_2\ldots B_n$ are in the same plane $\mathcal{P}$ and have the same center.

a) Show that $\prod_{j=1}^{n} B_iA_j = \prod_{i=1}^{n} A_jB_i, \forall i,j \in \{1,2,\ldots,n\}$.

b) Find $\min\limits_{M\in\mathcal{P}}\{MA_1 \cdot MA_2 \cdot \ldots \cdot MA_n + MB_1 \cdot MB_2 \cdot \ldots \cdot MB_n\}$.

43. Solve in $\mathbb{R}$ the equation $2^x + 2^{1-x} = 3a^{x(x-1)}$, where $a \geqslant 2$.

44. The base of a pyramid with apex at O is a cyclic polygon $A_1A_2\ldots A_n$, $n \geqslant 5$. A plane α intersects the edges OA_i at B_i, $i = 1,\ldots,n$. Prove that if the polygon $B_1B_2\ldots B_n$ is regular, then so is $A_1A_2\ldots A_n$.

PUTNAM SENIORS

45. The matrix $A \in \mathcal{M}_n(\mathbb{Z})$ is such that $\det A \neq 0$ and the equation $X^k = A$ has solutions in $\mathcal{M}_n(\mathbb{Z})$ for every $k \in \mathbb{N}^*$. Show that $A = I_n$.

46. For a permutation α of the set $[n] = \{1,2,\ldots,n\}$, let $\operatorname{Inv}\alpha$ denote the set of inversions of α. Given the permutations $\alpha_1, \alpha_2, \ldots, \alpha_N$ of $[n]$, show that

$$\sum_{k=0}^{N}|\operatorname{Inv}\alpha\alpha_k| \geqslant \frac{n(n-1)N}{4},$$

for some permutation α.

47. If $A \in \mathcal{M}_3(\mathbb{R})$ is a matrix such that $\operatorname{tr}(A - A^t)^{2n} = 0$ for some integer $n > 0$, show that $A = A^t$.

48. If $A, B \in \mathcal{M}_3(\mathbb{C})$ are non-singular matrices such that

$$A^2 - (\operatorname{tr} A)A + A^* = B^2 - (\operatorname{tr} B)B + B^*,$$

show that $A^* = B^*$.

49. If the matrices $A, B \in \mathcal{M}_3(\mathbb{Z})$ are singular, show that the number

$$\det(A^3 + B^3) + \det(A^3 - B^3)$$

is the double of a perfect cube.

50. Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be distinct positive numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that the equation

$$\prod_{i=1}^{n} a_i^{a_i^x} = \prod_{i=1}^{n} b_i^{b_i^x}$$

has at least one solution in $(0, \infty)$.

51. The sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are given by $x_0 \in [0,1]$, $x_{n+1} = x_n - nx_n^2$, $y_0 \in \mathbb{R}$, $y_{n+1} = 2y_n - y_n^2$. Define $(z_n)_{n\in\mathbb{N}}$ by $z_n = n^2 x_n + y_n$. Find all y_0 such that the sequence $(z_n)_{n\in\mathbb{N}}$ is convergent.

52. Let $f : (0,\infty) \to (0,\infty)$ be bounded function and let $p > 0$ be a real number. If $\lim\limits_{x\searrow 0}(f(x) - \alpha f^p(\alpha x)) = 0$, for every $\alpha > 0$, prove that

$$\lim_{x\searrow 0} f(x) = 0.$$

53. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the following property: for every $n \in \mathbb{N}^*$ there exists a number x_n such that $|f(x) - f(x_n)| \leqslant \frac{1}{n}$ for all $x > x_n$. Show that the f has a limit at infinity.

54. Let $a, b \in \mathbb{N}^*$. The sequence $(x_n)_{n\geqslant 2}$ is given by $x_2 > 0$ and

$$x_{n+1} = n^{\frac{1}{n^a}} x_n + n^{\frac{1}{n^b}} - 1, \quad \forall n \geqslant 2.$$

Show that the sequence is convergent if and only if $a, b \geqslant 2$.

55. Let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence of real numbers such that $\lim\limits_{n\to\infty} x_n = \infty$. Is it possible to find a bounded sequence $(y_n)_{n\in\mathbb{N}^*}$ such that the sequence given by

$$S_n = \sum_{k=1}^{n} \frac{x_n}{kx_n + y_n},$$

be bounded?

56. The sequences $(a_n)_{n\in\mathbb{N}^*}$ and $(b_n)_{n\in\mathbb{N}^*}$ satisfy $2 \leqslant a_1 < b_1$ and

$$a_{n+1} = a_n + \frac{b_n}{a_n} - 1, \quad b_{n+1} = b_n + \frac{a_n}{b_n} - 1,$$

for every $n \in \mathbb{N}^*$. Show both sequences converge to a_1.

57. Let a, b, c, d, e be real numbers such that $a > 0$, $e < 0$, $b^2 < \frac{8}{3}ac$ and

$$\frac{a}{2008} + \frac{b}{2007} + \frac{c}{2006} + \frac{d}{2005} + \frac{e}{2004} = 0.$$

Show that $a + b + c + d + e > 0$.

58. Let $a = 0{,}a_1a_2a_3\dots$, be the decimal representation of the real number $a \in (0,1)$.

a) Prove that the limit $f_a(x) = \lim\limits_{n\to\infty}(a_1x + a_2x^2 + \cdots + a_nx^n)$ exists and is finite, for every $x \in (0,1)$.

b) Prove that the function $f_a : (0,1) \to \mathbb{R}$ is rational if and only if a is rational.

59. The function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$f(0) = 0, \quad (1 + x^2)f'(x) \geqslant 1 + f^2(x),$$

for every $x \in \mathbb{R}$. Can f have a finite limit at infinity?

60. Let $f : (0,\infty) \to [0,\infty)$ be a bounded function. If

$$\lim_{x\to 0}\left(f(x) - \frac{1}{2}\sqrt{f\left(\frac{x}{2}\right)}\right) = 0 \quad \text{and} \quad \lim_{x\to 0}(f(x) - 2f^2(2x)) = 0,$$

show that $\lim\limits_{x\to 0} f(x) = 0$.

61. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ which have an antiderivative and satisfy the following condition: there exists $k \in \mathbb{N}^*$ such that $(\underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times } f})(x) = e^{-x}$, for all $x \in \mathbb{R}$.

62. Let $f : [0, \pi] \to \mathbb{R}$ be a continuous function such that

$$\lim_{n\to 0} \int_0^{\pi} |f(x) + a_n \cos x + b_n \cos 2x| = 0,$$

for two conveniently chosen sequences $(a_n)_n$, $(b_n)_n$. Show that

$$\int_0^{\pi} f(x)\mathrm{d}x = 0.$$

63. Let $f : [0, 1] \to \mathbb{R}$ be a continuous concave function with $f(0) = 1$. Show that

$$\frac{p+2}{2} \int_0^1 x^p f(x)\mathrm{d}x \leqslant \int_0^1 f(x)\mathrm{d}x - \frac{p}{2(p+1)}, \quad \forall p \geqslant 1.$$

When does equality hold?

64. Let $G_1, G_2, \ldots, G_n$ be groups and $G = G_1 \times G_2 \times \cdots \times G_n$ their direct product.

a) If $f : (\mathbb{Q}, +) \to G$ is an injective morphism, show that there exist k, $1 \leqslant k \leqslant n$ and an injective morphism $g_k : (\mathbb{Q}, +) \to G_k$.

b) Does the above statement still hold if G is the direct product of countably many groups $G_1, G_2, \ldots, G_n, \ldots$?

65. Let $f : (0, \infty) \to (0, 1)$ be a continuous decreasing function and $(a_n)_{n\geqslant 1}$ be a strictly increasing positive sequence, such that the sequence $(\frac{a_{n+1}}{a_n})_{n\geqslant 1}$ is strictly decreasing. Consider the sequence $(I_n)_{n\geqslant 1}$, where

$$I_n = \frac{1}{a_n} \int_{a_n}^{a_{n+1}} f(x)\mathrm{d}x, \quad \forall n \geqslant 1.$$

a) Show that the sequence $(I_n)_{n\geqslant 1}$ is decreasing.

b) Find $\lim_{n\to\infty} I_n$ in the case $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

66. Let $(G, \cdot)$ be a commutative group with n elements and $a \in G$. Find the number of the functions $f : G \to G$ satisfying

$$\prod_{x\in G} f(x) = a.$$

67. Let $f, f_n, g_n : [0, 1] \to \mathbb{R}, n \geqslant 1$ be functions such that

$$|f(x) - f(y)| \leqslant |x - y|, \quad \forall x, y \in [0, 1],$$

$$f_1 = f, \quad f_{n+1}(x) = \int_0^1 f_n(tx)\mathrm{d}t, \quad \forall x \in [0, 1], \quad \forall n \in \mathbb{N}^*$$

$$g_n(x) = \sum_{k=1}^{n} f_k(x), \forall x \in [0, 1], \quad \forall n \in \mathbb{N}^*.$$

Prove that if there exists $a \in [0, 1]$ such that the sequence $(g_n(a))_{n\geqslant 1}$ is convergent, then the sequence $(g_n(x))_{n\geqslant 1}$ is convergent for every $x \in [0, 1]$.

68. Prove that the equation

$$\frac{x^{2008}}{2008} + \frac{x^{2007}}{2007} + \frac{x^{2006}}{2006} + \cdots + \frac{x^2}{2} + x = \frac{1}{2}, \quad x > 0$$

has an unique solution x_0 and

$$\left|1 - \frac{1}{\sqrt{e}} - x_0\right| < 10^{-500}.$$

69. Let $f : [0, 1] \to \mathbb{R}$ be an integrable function such that

$$\lim_{n\to\infty} n \int_x^{x+\frac{1}{n}} f(t)\mathrm{d}t = 0, \quad \forall x \in [0, 1).$$

Prove that

$$\int_a^b f(t)\mathrm{d}t = 0, \quad \forall a, b \in (0, 1).$$

70. Prove that

$$\lim_{n\to\infty} \int_1^n \frac{n2^n}{(1 + x^2)^n} \mathrm{d}x = 1.$$

71. Consider the sequence $(a_n)_{n\in\mathbb{N}^*}$, where

$$a_n = \frac{3}{2} \ln n - \int_0^1 \ln\left(1^t + 2^t + \cdots + n^t\right) \mathrm{d}t.$$

a) Show that the sequence $(a_n)_{n\in\mathbb{N}^*}$ is convergent and evaluate its limit a.

b) Show that $0 < n(a - a_n) < 3/2$, for every $n \in \mathbb{N}^*$.

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PART TWO

SOLUTIONS

PROBLEMS AND SOLUTIONS

DISTRICT ROUND

7$^{\text{th}}$ GRADE

Problem 1. Show that

$$n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) \geqslant (n+1)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right),$$

for all natural numbers $n \geqslant 1$.

Solution. Put $x = \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. We have $n(1+x) \leqslant (n+1)(x+\frac{1}{n+1}) \Leftrightarrow n \leqslant x+1$. As $x+1 = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \leqslant 1+1+\cdots+1 = n$, this concludes the proof.

Problem 2. Consider the square $ABCD$ and the point E on the side AB. The diagonal AC intersects the segment DE at point P. The perpendicular from point P to DE intersects side BC at point F. Prove that $EF = AE + FC$.

Solution. Extend the segment BC by $CQ = AE$, such that point C is between Q and B. Triangles DAE and DCQ are then equal, so $DE = DQ$ and $\angle QDC = \angle ADE$. Thus $\angle EDQ = 90°$ and we get $\angle EDF = \angle PCF = 45°$. Then $\angle FDQ = \angle EDQ - \angle EDF = 45°$ and triangles DEF and DFQ are equal. Finally $EF = FQ = FC + CQ = AE + FC$, concluding thus the proof.

Problem 3. In a school there are 10 classrooms. Each student in a classroom knows exactly one student in each of the other 9 classrooms. Prove that the number of students in each classroom is the same.

(Assume that if student A knows student B, then student B knows student A too.)

Solution. Fix two arbitrary classes, say X and Y. It is suficient to prove that X and Y have the same number of students. If not so, suppose that the number of students in X is larger than that in Y.

As any student in X knows exactly one student in Y, by the PHP there will be two students, call them A and B that know the same student C in Y. But then C knows A and B, that is two students in X, contradicting thus the hypothesis.

Problem 4. Let $M = \{1, 2, 4, 5, 7, 8, \ldots\}$ be the set of natural numbers not divisible by 3. The sum of $2n$ consecutive elements of set M is 300. Determine the possible values of n.

Solution. The first of the $2n$ number has the form $3k+1$ or $3k+2$, so we have two possible cases:

i) The numbers are $3k+1, 3k+2, 3k+4, 3k+5, \ldots, 3k+3n-2, 3k+3n-1$. Their sum is $(6k+3) + (6k+9) + (6k+15) + \cdots + (6k+6n-3) = 6kn + 3(1+3+5+\cdots+2n-1) = 6kn + 3n^2 = 3n(2k+n)$, so $n(2k+n) = 100$. Since both factors have the same parity and the second is larger, either $n = 2$ or $n = 10$.

ii) The numbers are $3k+2, 3k+4, 3k+5, \ldots, 3k+3n-2, 3k+3n-1, 3k+3n+1$. Their sum is by $3n$ larger then in the preceeding case, so $3n(2k+n)+3n = 3(2k+n+1)$. We get $n(2k+n+1) = 100$. This time the two left factors have opposite parities, and again the second is larger. The only possibilities are $n = 1$, $n = 4$ and $n = 5$.

8th GRADE

Problem 1. If the intersection of a regular tetrahedron and a plane is a rhombus, prove that the rhombus must be a square.

Solution. Two opposite edges of the rhombus are parallel. The faces containing these edges meet on an edge of the tetrahedron; for example AB, which in turn is parallel to the section plane. In the same way the edge CD is parallel with the section plane. As $AB \perp CD$ we get that two consecutive rhombus edges are perpendicular. This means that the rhombus is a square.

Problem 2. Determine the irrational numbers x such that both $x^2 + 2x$ and $x^3 - 6x$ are rational numbers.

Solution. We have $a = x^2 + 2x + 1 = (x+1)^2 \in \mathbb{Q}$, where $a \geqslant 0$, thus $x = -1 \pm \sqrt{a}$. Then $x^3 - 6x = -1 \pm 3\sqrt{a} - 3a \pm a\sqrt{a} + 6 \mp 6\sqrt{a} = 5 - 3a \pm (a-3)\sqrt{a}$. As $x^3 - 6x$ and $5 - 3a$ are rational numbers we deduce that $(3-a)\sqrt{a} \in \mathbb{Q}$. If $a \neq 3$, then $\sqrt{a} \in \mathbb{Q}$ and consequently $x \in \mathbb{Q}$, a contradiction. We get $a = 3$ and thus $x = -1 \pm \sqrt{3}$.

Alternatively, let $m = x^2 + 2x$, $n = x^3 - 6x$, $m, n \in \mathbb{Q}$. We have $n = x^3 + 2x^2 - 2x^2 - 4x - 2x = mx - 2x - 2m = (m-2)x - 2m$, where $x \notin \mathbb{Q}$. We get $n = -2m$ and $m - 2 = 0$, so $m = 2$, $x^2 + 2x - 2 = 0$ and $x = -1 \pm \sqrt{3}$.

Problem 3. Consider the cube $ABCDA'B'C'D'$ and the points M, N, P, such that M is the foot of the perpendicular from A to plane $(A'CD)$, N is the foot of the perpendicular from B to the diagonal $A'C$, and P is the symmetric of D with respect to C. Show that the points M, N, P are collinear.

Solution. Let a be the length of the cube edge. Point M is the midpoint of $A'D$. We have $CN = \frac{BC^2}{A'C} = \frac{a^2}{a\sqrt{3}} = \frac{a}{\sqrt{3}} = \frac{A'C}{3}$, and points M, N, P belong to the plane $(A'CD)$.

By Menelaos theorem in triangle $A'CD$, we need to show that $\frac{DP}{PC} \cdot \frac{CN}{NA'} \cdot \frac{A'M}{MD} = 1$. As $\frac{DP}{PC} = 2$, $\frac{CN}{NA'} = \frac{1}{2}$ and $\frac{A'M}{MD} = 1$, the above relation is fulfilled.

Problem 4. Determine the strictly positive real numbers x, y, z that satisfy simultaneously the conditions: $x^3y + 3 \leqslant 4z$, $y^3z + 3 \leqslant 4x$, and $z^3x + 3 \leqslant 4y$.

Solution. Multiply the three inequalities $x^3y \leqslant 4z - 3$, $y^3z \leqslant 4x - 3$ and $z^3x \leqslant 4y - 3$, to get $x^4y^4z^4 \leqslant (4x-3)(4y-3)(4z-3)$. On the other hand, by the AM-GM inequality $x^4 + 3 = (x^4+1) + 2 \geqslant 2x^2 + 2 = 2(x^2+1) \geqslant 4x$, that is $x^4 \geqslant 4x - 3$, where equality holds iff $x = 1$. Multiplying $x^4 \geqslant 4x - 3$, $y^4 \geqslant 4y - 3$, $z^4 \geqslant 4z - 3$ yields $x^4y^4z^4 \geqslant (4x-3)(4y-3)(4z-3)$, so $x = y = z = 1$, by the equality case.

9th GRADE

Problem 1. Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers such that $|a_{n+1}-a_n|\leqslant 1$, for all $n\in\mathbb{N}^*$, and $(b_n)_{n\geqslant 1}$ the sequence defined by

$$b_n=\frac{a_1+a_2+\cdots+a_n}{n}.$$

Show that $|b_{n+1}-b_n|\leqslant \frac{1}{2}$, for all $n\in\mathbb{N}^*$.

Solution. Since $|a_{n+1}-a_n|\leqslant 1$, the triangle inequality yields $|a_n-a_m|\leqslant |n-m|$, for all $n,m\in\mathbb{N}^*$, so

$$|b_{n+1}-b_n|=\left|\frac{a_1+\cdots+a_{n+1}}{n+1}-\frac{a_1+\cdots+a_n}{n}\right|=\frac{|na_{n+1}-a_1-\cdots-a_n|}{n(n+1)}$$
$$=\frac{|a_{n+1}-a_1+\cdots+a_{n+1}-a_n|}{n(n+1)}\leqslant\frac{|a_{n+1}-a_1|+\cdots+|a_{n+1}-a_n|}{n(n+1)}$$
$$\leqslant\frac{n+(n-1)+\cdots+1}{n(n+1)}=\frac{1}{2}.$$

The conclusion follows.

Problem 2. Consider the set $A=\{1,2,\ldots,n\}$, where $n\in\mathbb{N}$, $n\geqslant 6$. Show that A is the union of three pairwise disjoint sets, with the same cardinality and the same sum of their elements, if and only if n is a multiple of 3.

Solution. If $A=B\cup C\cup D$, where $B\cap C=C\cap D=D\cap B=\emptyset$ and $|B|=|C|=|D|$, then $|A|=3|B|$, so $3|n$.

For the converse, let $n\in\mathbb{N}$, $n\geqslant 6$, n divisible by 3. Two cases may occur:

i) For $n=6p$, $p\in\mathbb{N}^*$, the sets

$$B=\{6k+1,\ 6k+6\mid k=0,1,\ldots,p-1\},$$
$$C=\{6k+2,\ 6k+5\mid k=0,1,\ldots,p-1\},$$
$$D=\{6k+3,\ 6k+4\mid k=0,1,\ldots,p-1\},$$

form a partition of A with the required properties.

ii) For $n=6p+3$, $p\in\mathbb{N}^*$, we have the following two possibilities:

a) if $p=1$ let $B=\{1,5,9\}$, $C=\{2,6,7\}$, $D=\{3,4,8\}$;

b) if $p\geqslant 2$, consider

$$B=\{1,5,9\}\cup\{6k+10,6k+15\mid k=0,1,\ldots,p-2\},$$
$$C=\{2,6,7\}\cup\{6k+11,6k+14\mid k=0,1,\ldots,p-2\},$$
$$D=\{3,4,8\}\cup\{6k+12,6k+13\mid k=0,1,\ldots,p-2\}.$$

Problem 3. Show that if $n\geqslant 4$, $n\in\mathbb{N}$ and $\left[\frac{2^n}{n}\right]$ is a power of 2, then n is a power of 2.

Solution. If $\left[\frac{2^n}{n}\right]=2^k$, where $k\in\mathbb{N}$, then $n2^k\leqslant 2^n<n(2^k+1)$, so there is $r\in\mathbb{N}$, $0\leqslant r\leqslant n-1$ such that $2^n=n2^k+r$. The following cases can arise:

i) If $2^k<n$, then $2^n<n^2+n$. By easy induction $2^n>n^2+n$ for $n\geqslant 5$. Clearly $n=4$ works.

ii) For $n\geqslant 5$ we get $2^k\geqslant n$, or $2^k>r$. But $2^n=n2^k+r$ implies $2^k|r$, so $r=0$ and $n=2^{n-k}$ is a power of 2.

Problem 4. Let $ABCD$ be a quadrilateral that can be inscribed in a circle. Denote by P the intersection point of lines AD and BC, and by Q the intersection point of lines AB and CD. Let E be the fourth vertex of the parallelogram $ABCE$, and F the intersection of lines CE şi PQ. Prove that the points D,E,F, and Q lie on the same circle.

Solution. Since $\angle BAP=\angle DCP$, triangles BAP and DCP are similar, so

$$\frac{BA}{DC}=\frac{BP}{DP}. \tag{1}$$

We have $\angle CDP=\angle CBQ$ (or $\angle CDP=180^\circ-\angle CBQ$) and $\angle BCQ$ (or $\angle PCD=180^\circ-\angle BCQ$). By the sine theorem in triangles BCQ, DCP we obtain

$$\frac{CP}{DP}=\frac{CQ}{BQ}. \tag{2}$$

Moreover,

$$\frac{CF}{BQ}=\frac{CP}{BP} \tag{3}$$

and $EC=AB$. (4)

Consequently,

$$EC \cdot CF = \frac{AB \cdot CP \cdot BQ}{BP}$$

by (3) and (4), which in turn equals $\frac{DC \cdot CP \cdot BQ}{DP}$ by (1). The latter equals $DC \cdot CQ$ by (2), so D, E, F, Q are concyclic.

10$^{\text{th}}$ GRADE

Problem 1. Let a and b be two complex numbers. Prove the inequality

$$|1+ab| + |a+b| \geqslant \sqrt{|a^2-1| \cdot |b^2-1|}.$$

Solution. By the triangle inequality

$$|1+ab| + |a+b| \geqslant |1+ab+a+b|$$

and

$$|1+ab| + |a+b| \geqslant |1+ab-a-b|.$$

Multiply the last two to get

$$(|1+ab| + |a+b|)^2 \geqslant |(1+ab)^2 - (a+b)^2|$$

which is equivalent to $|1+ab| + |a+b| \geqslant \sqrt{|a^2-1| \cdot |b^2-1|}$.

Alternative solution. We have

$$|1+2ab+a^2b^2| + |a^2+2ab+b^2| \geqslant |a^2b^2+1-a^2-b^2| = |a^2-1| \cdot |b^2-1|$$

which is equivalent to

$$(|1+ab| + |a+b|)^2 \geqslant |(1+ab)^2 - (a+b)^2|.$$

Problem 2. Determine the integers x such that

$$\log_3(1+2^x) = \log_2(1+x).$$

Solution. We shall prove that $x = 1$ or $x = 3$. Obviously, x is a non-negative integer. By inspection, 1 and 3 are solutions and $\{0, 2, 4, 5\}$ are not. Write the given equation in the equivalent form $1 + 2^x = (1+x)^{\log_2 3}$.

For $x \geqslant 6$ we get $1 + 2^x > (1+x)^2$ by induction. Since $2 > \log_2 3$, there are no other solutions.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{2}, \quad \text{for all } x, y \in \mathbb{R}.$$

a) Prove that the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = f(x) - f(0)$ is additive, i.e. $g(x+y) = g(x) + g(y)$, for all $x, y \in \mathbb{R}$.

b) Show that f is a constant function.

Solution. a) Let $x, y \in \mathbb{R}$; the given equality yields:

$$\begin{aligned}\frac{g(x)+g(y)}{2} &= \frac{f(x)+f(y)}{2} - f(0) = f\left(\frac{x+y}{3}\right) - f(0)\\ &= \frac{f(x+y)-f(0)}{2} - f(0) = \frac{g(x+y)}{2},\end{aligned}$$

which means $g(x+y) = g(x) + g(y)$, for any $x, y \in \mathbb{R}$.

b) For $x = y$ we obtain $g(2x) = 2g(x)$, for any $x \in \mathbb{R}$. The property of f also implies $g(\frac{x+y}{3}) = \frac{g(x)+g(y)}{2}$ so $g(\frac{2x+x}{3}) = \frac{g(2x)+g(x)}{2}$, for any $x \in \mathbb{R}$. So, $g(2x) = g(x)$ and consequently $g(x) = 0$ for any $x \in \mathbb{R}$. Thus $f(x) = f(0)$ for all $x \in \mathbb{R}$, so f is a constant function.

Problem 4. Let $n \geqslant 3$ be an integer and $z = \cos\frac{2\pi}{n} + \mathrm{i}\sin\frac{2\pi}{n}$. Consider the sets

$$A = \{1, z, z^2, \ldots, z^{n-1}\}$$

and

$$B = \{1, 1+z, 1+z+z^2, \ldots, 1+z+\cdots+z^{n-1}\}.$$

Determine the set $A \cap B$.

Solution. Clearly, $1 \in A \cap B$. Let $w \in A \cap B$, $w \neq 1$. As a member of B, $w = 1 + z + \cdots + z^k = \frac{1-z^{k+1}}{1-z}$ for some $k =, 1, 2, \ldots, n-1$. Since $w \in A$, we get $|w| = 1$, and $|1 - z^{k+1}| = |1 - z|$.

The latter equality yields $\sin\frac{(k+1)\pi}{n} = \sin\frac{\pi}{n}$, or $\frac{(k+1)\pi}{n} = \pi - \frac{\pi}{n}$, which implies $k = n-2$, so $w = \frac{1-\frac{1}{z}}{1-z} = -\frac{1}{z}$. Because $w \in A$, we must have $w^n = 1$, which means that n has to be even.

So, the answer is: $A \cap B = \{1\}$ for odd n, and $A \cap B = \{1, -\frac{1}{z}\}$ for even n.

11th GRADE

Problem 1. If $A \in \mathcal{M}_2(\mathbb{R})$, show that

$$\det(A^2 + A + I_2) \geqslant \frac{3}{4}(1 - \det A)^2.$$

Solution. If $p(X) = \det(A - XI_2) = X^2 - aX + b$ is the characteristic polynomial of A then

$$\det(A^2 + A + I) = \det(A - \varepsilon I_2)(A - \varepsilon^2 I_2) = p(\varepsilon)p(\varepsilon^2),$$

where ε is a primitive cube root of unity. It follows that

$$\det(A^2 + A + I) = (\varepsilon^2 - a\varepsilon + b)(\varepsilon - a\varepsilon^2 + b) = a^2 + a(b+1) + b^2 - b + 1.$$

The minimum value of the quadratic function $a^2 + a(b+1) + b^2 - b + 1$ is attained at $a = -(b+1)/2$, and equals $\frac{3}{4}(1-b)^2$. Since $b = \det A$, the conclusion follows.

Problem 2. Consider $A, B \in \mathcal{M}_n(\mathbb{R})$. Show that $\operatorname{rank} A + \operatorname{rank} B \leqslant n$ if and only if there exists an invertible matrix $X \in \mathcal{M}_n(\mathbb{R})$, such that $AXB = O_n$.

Solution. Suppose $AXB = O_n$ for some invertible matrix X. Applying Sylvester's inequality we get

$$\begin{aligned} 0 = \operatorname{rank}(AXB) &\geqslant \operatorname{rank}(AX) + \operatorname{rank} B - n \\ &\geqslant \operatorname{rank} A + \operatorname{rank} X + \operatorname{rank} B - 2n \\ &= \operatorname{rank} A + \operatorname{rank} B - n. \end{aligned}$$

Conversely, suppose that $\operatorname{rank} A = a$, $\operatorname{rank} B = b$ and $a + b \leqslant n$. We can find invertible matrices P, Q, R, S such that

$$PAQ = \begin{pmatrix} I_a & O_{a,n-a} \\ O_{n-a,a} & O_{n-a,n-a} \end{pmatrix}, \quad RBS = \begin{pmatrix} O_{n-b,n-b} & O_{n-b,b} \\ O_{b,n-b} & I_b \end{pmatrix}.$$

It follows that $PAQRBS = O_n$, whence $AQRB = O_n$. We can now set $X = QR$ to get the desired conclusion.

Alternative solution. The existence of an invertible matrix X so that $AXB = O_n$ is equivalent to the existence of an invertible linear transformation on $\mathbb{R}^n$ sending $\operatorname{Im} B$ to $\operatorname{Ker} A$. Such a linear transformation exists iff $\dim(\operatorname{Im} B) \leqslant \dim(\operatorname{Ker} A)$. Since $\dim(\operatorname{Im} B) = \operatorname{rank} B = b$ and $\dim(\operatorname{Ker} A) = n - \operatorname{rank} A = n - a$, the conclusion follows at once.

Problem 3. Let $(x_n)_{n\geqslant 1}$, $(y_n)_{n\geqslant 1}$ be two sequences of strictly positive numbers, such that

$$x_{n+1} \geqslant \frac{x_n + y_n}{2}, \quad y_{n+1} \geqslant \sqrt{\frac{x_n^2 + y_n^2}{2}}, \quad \text{for all } n \in \mathbb{N}^*.$$

a) Show that the limits of the sequences $(x_n + y_n)_{n\geqslant 1}$ and $(x_n y_n)_{n\geqslant 1}$ exist.

b) Show that the limits of the sequences $(x_n)_{n\geqslant 1}$, $(y_n)_{n\geqslant 1}$ exist and are equal.

Solution. Define the sequences s_n, p_n by $s_n = x_n + y_n$, $p_n = x_n y_n$ for all $n \geqslant 1$.

For part a) we have $x_{n+1} \geqslant \frac{1}{2}s_n \geqslant \sqrt{p_n}$, $y_{n+1} \geqslant \frac{1}{2}s_n \geqslant \sqrt{p_n}$, whence $s_{n+1} \geqslant s_n$, $p_{n+1} \geqslant p_n$. It follows that the sequences $(s_n)_{n\geqslant 1}$ and $(p_n)_{n\geqslant 1}$ are nondecreasing and therefore have a limit.

For part b), if $s_n \to \infty$, we get, using the above inequalities, that $x_n \to \infty$ and $y_n \to \infty$.

If $s_n \to s \in \mathbb{R}$, then the sequence $(p_n)_{n\geqslant 1}$ is bounded above by $\frac{1}{4}s^2$, so it converges to some $p \leqslant \frac{1}{4}s^2$. We also have $p_{n+1} = x_{n+1}y_{n+1} \geqslant \frac{1}{4}s_n^2$ whence, passing to the limit, we obtain the reverse inequality.

Since $x_n, y_n \in \{\frac{1}{2}(s_n \pm \sqrt{s_n^2 - 4p_n})\}$, it follows that

$$\left|x_n - \frac{1}{2}s_n\right| = \left|y_n - \frac{1}{2}s_n\right| = \frac{1}{2}\left|\sqrt{s_n^2 - 4p_n}\right| \to 0,$$

so $x_n \to \frac{1}{2}s$ and $y_n \to \frac{1}{2}s$.

Problem 4. Determine for what values of $a \in [0, \infty)$ there exist continuous functions $f : \mathbb{R} \to \mathbb{R}$, such that

$$f(f(x)) = (x - a)^2, \quad \text{for all } x \in \mathbb{R}.$$

Solution. If $a = 0$ we can take $f(x) = |x|^{\sqrt{2}}$ and we have $f(f(x)) = x^2$, as desired.

Assume now that $a \neq 0$. If $f(x) = f(y)$ for $x \neq y$ we get $(x-a)^2 = (y-a)^2$, whence $x - a = a - y$. It follows that the restrictions of f to $[a, \infty)$ and $(-\infty, a]$ are both one-to-one, hence strictly monotonic. Since f is not monotonic over the whole real line, and since f is not bounded from above, it follows that f is decreasing on $(-\infty, a]$ and increasing on $[a, \infty)$.

Therefore, $f([a, \infty)) = f((-\infty, a]) = [f(a), \infty)$ and a is the unique global minimum point of f. We thus have $f(a) \leqslant f(f(a)) = 0$. If $f(a) = 0 = f(f(a))$, we get, by uniqueness of a, that $a = f(a) = 0$. Suppose now that $f(a) < 0$. By continuity, we can find $b > a$ such that $f(b) < 0$. Since $b > 0 > f(a)$, we can find $c \in (a, \infty)$ such that $b = f(c)$, whence $(c - a)^2 = f(f(c)) = f(b) < 0$, a contradiction.

12th GRADE

Problem 1. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)\mathrm{d}x = \int_0^1 x f(x)\mathrm{d}x$. Show that there exists $c \in (0, 1)$ such that

$$f(c) = \int_0^c f(x)\mathrm{d}x.$$

Solution. Let $F : [0, 1] \to \mathbb{R}$, $F(x) = \int_0^x f(t)\mathrm{d}t$. We have

$$\begin{aligned} F(1) &= \int_0^1 f(x)\mathrm{d}x = \int_0^1 x f(x)\mathrm{d}x = \int_0^1 x F'(x)\mathrm{d}x \\ &= xF(x)\Big|_0^1 - \int_0^1 F(x)\mathrm{d}x = F(1) - \int_0^1 F(x)\mathrm{d}x, \end{aligned}$$

thus $\int_0^1 F(x)\mathrm{d}x = 0$.

Consider the map $g : [0, 1] \to \mathbb{R}$, defined by $g(x) = \mathrm{e}^{-x} \int_0^x F(t)\mathrm{d}t$. We have $g(0) = 0 = g(1)$, so we can apply Rolle's theorem to get some $b \in (0, 1)$ for which $g'(b) = 0$, or equivalently

$$F(b) = \int_0^b F(x)\mathrm{d}x.$$

We can now apply Rolle's theorem to the map $G : [0, b] \to \mathbb{R}$, defined by $G(x) = F(x) - \int_0^x F(t)\mathrm{d}t$ to deduce the existence of an element $c \in (0, b)$ for which $G'(c) = 0$, that is

$$f(c) = \int_0^c f(x)\mathrm{d}x,$$

which is what we wanted to prove.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, periodic function of period T. If F is an antiderivative of f, show that:

a) the function $G : \mathbb{R} \to \mathbb{R}$ given by

$$G(x) = F(x) - \frac{x}{T}\int_0^T f(t)\mathrm{d}t$$

is periodic;

b)

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{F(k)}{n^2 + k^2} = \frac{\ln\sqrt{2}}{T}\int_0^T f(x)\mathrm{d}x.$$

Solution. a) Since $F' = f$, we can write $F(x) = \int_0^x f(t)\mathrm{d}t + c$ for some constant $c \in \mathbb{R}$. We then have

$$\begin{aligned} G(x + T) &= \int_0^{x+T} f(t)\mathrm{d}t + c - \frac{x+T}{T}\int_0^T f(t)\mathrm{d}t \\ &= \int_0^x f(t)\mathrm{d}t + \int_x^{x+T} f(t)\mathrm{d}t + c - \frac{x}{T}\int_0^T f(t)\mathrm{d}t - \int_0^T f(t)\mathrm{d}t \\ &= \int_0^x f(t)\mathrm{d}t + c - \frac{x}{T}\int_0^T f(t)\mathrm{d}t \\ &= F(x) - \frac{x}{T}\int_0^T f(t)\mathrm{d}t = G(x), \end{aligned}$$

so G is periodic.

b) G is bounded on $\mathbb{R}$, since it is continuous and periodic. We can therefore consider $M = \max\{|G(x)| : x \in \mathbb{R}\}$. We have

$$\left| \sum_{k=1}^{n} \frac{G(k)}{n^2 + k^2} \right| \leqslant M \cdot \left| \sum_{k=1}^{n} \frac{1}{n^2 + k^2} \right| \leqslant M \cdot n \cdot \frac{1}{n^2} = \frac{M}{n},$$

whence

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{F(k)}{n^2+k^2}=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{G(k)}{n^2+k^2}+\left(\frac{1}{T}\int_0^T f(x)\mathrm{d}x\right)\sum_{k=1}^{n}\frac{k}{n^2+k^2}\right)$$
$$=\left(\frac{1}{T}\int_0^T f(x)\mathrm{d}x\right)\cdot\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2}$$
$$=\left(\frac{1}{T}\int_0^T f(x)\mathrm{d}x\right)\cdot\left(\int_0^1\frac{x}{1+x^2}\mathrm{d}x\right)$$
$$=\frac{\ln\sqrt{2}}{T}\int_0^T f(x)\mathrm{d}x.$$

Problem 3. Let A be a commutative unitary ring with an odd number of elements. If n is the number of solutions to the equation $x^2=x$, $x\in A$, and m is the number of invertible elements in A, show that n divides m.

Solution. Consider the set $I(A)=\{x : x\in A,\ x^2=x\}$. If $x\in I(A)$, then $(2x-1)^2=4x^2-4x+1=1$, so $2x-1\in U(A)$ (here $U(A)$ denotes the group of units of the ring A). Since $|A|$ is odd, $2=1+1$ is a unit of A, and therefore the map $f: I(A)\to U(A)$, defined by $f(x)=2x-1$, is one-to-one. This shows that $|I(A)|=|\mathrm{Im}\, f|$, so to complete the proof it suffices to show that $\mathrm{Im}\, f$ is a subgroup of $U(A)$ and then appeal to Lagrange's theorem. Since $U(A)$ is finite, it suffices to prove that $uv\in\mathrm{Im}\, f$ whenever $u,v\in\mathrm{Im}\, f$.

Let $u,v\in\mathrm{Im}\, f$, $u=2x-1$, $v=2y-1$ for some $x,y\in I(A)$. If we let $z=2^{-1}(uv+1)$, then $uv=2z-1$, so we are done once we prove that $z\in I(A)$, i.e. $z^2=z$, or equivalently (since 4 is a unit) $4z^2=4z$. We have

$$4z^2=(1+uv)^2=1+2uv+u^2v^2=2(1+uv)=4z,$$

and the conclusion follows.

REMARK. We can say a little more about the order of $I(A)$: since $\mathrm{Im}\, f$ is contained in the subgroup of $U(A)$ consisting of elements of order 2, which is a 2-group, it follows that $\mathrm{Im}\, f$ is also a 2-group, hence $|I(A)|$ is a power of 2.

Problem 4. Let K be a finite field. We say that two polynomials f and g in $K[X]$ are *neighbours* if they have the same degree and they differ by exactly one coefficient.

a) Show that all neighbours of the polynomial $X^2+\hat{1}\in\mathbb{Z}_3[X]$ can be factored out in $\mathbb{Z}_3[X]$.

b) If the number of elements in K is $q\geqslant 4$, show that any polynomial of degree $q-1$ in $K[X]$ has a neighbour that can be factored out in $K[X]$ and also has a neighbour with no roots in K.

Solution. a) The neighbours of X^2+1 are: $2X^2+1$, X^2+X+1, X^2+2X+1, X^2 and X^2+2. They all have roots in $\mathbb{Z}_3$, and are therefore reducible in $\mathbb{Z}_3[X]$.

b) Let $f\in K[X]$ having $\deg f=q-1$. We shall use the same notation for a polynomial and its corresponding polynomial function. If $f(0)=0$, we choose $g=f+X$, otherwise we set $g=f-f(0)$. In both cases g is a neighbour of f which is clearly reducible since it has 0 as a root.

Recall that

$$\sum_{\alpha\in K}\alpha^s=\begin{cases}0 & \text{if } (q-1)\nmid s,\\ -1 & \text{if } (q-1)\mid s,\ s\geqslant 1.\end{cases}$$

Write f as $\sum_{i=0}^{q-1}a_iX^i$, $a_i\in K$. We distinguish two cases:

Case 1: $a_0=0$. Since

$$\sum_{\alpha\in K}f(\alpha)=\sum_{\alpha\in K}\sum_{i=0}^{q-1}a_i\alpha^i=\sum_{i=0}^{q-1}a_i\left(\sum_{\alpha\in K}\alpha^i\right)=-a_{q-1}\neq 0,$$

it follows that f is not onto (otherwise $\sum_{\alpha\in K}f(\alpha)=\sum_{\alpha\in K}\alpha=0$). Since $0\in\mathrm{Im}\, f$ and f is not onto, we can find a nonzero element $s\notin\mathrm{Im}\, f$. The polynomial $g=f-s$ is then a neighbour of f with no roots in K.

Case 2: $a_0\neq 0$. If $a_i=0$ for $i\notin\{0,q-1\}$, then $\mathrm{Im}\, f=\{a_0,a_{q-1}+a_0\}$. Since $|K|\geqslant 4$, we can find a nonzero $s\notin\mathrm{Im}\, f$ and $g=f-s$ is again a neighbour of f without roots in K.

Assume now that $a_i\neq 0$ for some $i\in\{1,2,\ldots,q-2\}$ and consider the polynomial $h=X^{q-i-1}f$. Since

$$\sum_{\alpha\in K}h(\alpha)=\sum_{j=0}^{q-1}a_j\left(\sum_{\alpha\in K}\alpha^{q-i+j-1}\right)=-a_i\neq 0,$$

it follows as before that h is not onto, so we can find a nonzero $s\notin\mathrm{Im}\, h$. The polynomial $g=f-sX^i$ is then a neighbour of f and we must check that it has no roots in K. Clearly, 0 is not a root of g since $g(0) = f(0) \neq 0$. If $\alpha \in K^*$ is a root of g then

$$s = \alpha^{q-1}s = \alpha^{q-i-1}(s\alpha^i) = \alpha^{q-i-1}(f(\alpha) - g(\alpha)) = \alpha^{q-i-1}f(\alpha) = h(\alpha),$$

contradicting the fact that s was chosen outside $\operatorname{Im} h$.

PROBLEMS AND SOLUTIONS

FINAL ROUND

7$^{\text{th}}$ GRADE

Problem 1. The acute triangle ABC has $B > C$. Consider the altitude AD, $D \in BC$, and the perpendicular DE to AC, $E \in AC$. Consider the point F on the segment DE. Prove that the lines AF and BF are perpendicular if and only if $EF \cdot DC = BD \cdot DE$.

Solution. If $AF \perp BF$ notice that the quadrilateral $ABDF$ is circular, so $\angle ABF = \angle ADF$ and $\angle BAF = \angle DAE$. Consequently, $\angle BAD = \angle FAE$, which proves that triangles ABD and AFE are similar, so $\frac{BD}{FE} = \frac{AD}{AE}$.

On the other hand, triangles ΔADE and DCE are similar, so $\frac{AD}{AE} = \frac{CD}{DE}$. The last two equalities yields $\frac{BD}{FE} = \frac{CD}{DE}$, that is, $EF \cdot DC = BD \cdot DE$.

Conversely, if $EF \cdot DC = BD \cdot DE$ we get $\frac{BD}{FE} = \frac{CD}{DE}$ which implies that triangles ADE and DCE are similar, so $\frac{CD}{DE} = \frac{AD}{AE}$. In the same way, $\frac{BD}{FE} = \frac{AD}{AE}$ implies that triangles ABD and AFE are similar and, as a consequently, $\angle AFE = \angle ABD$. Thus the quadrilateral $ABDF$ is circular, so $\angle AFB = \angle BDA$. This implies that $AF \perp BF$.

Problem 2. Given that a rectangle can be divided into 200 and into 288 equal squares, prove that it can also be divided into 392 equal squares.

Solution. Divide the sides of the rectangle, say of length a (respectively b), into m_1 (respectively n_1) segments of length u to get 200 squares of side length u and into m_2 (respectively n_2) segments of length v to get 288 squares of side length v. We have $m_1n_1 = 200$, $m_2n_2 = 288$. Moreover, $\frac{a}{m_1} = \frac{b}{n_1} = u$ and $\frac{a}{m_2} = \frac{b}{n_2} = v$. Thus $u^2 = \left(\frac{a}{m_1}\right)^2 = \frac{ab}{200}$, $v^2 = \left(\frac{a}{m_2}\right)^2 = \frac{ab}{288}$. This implies $m_1^2 = \frac{200a}{b}$, $m_2^2 = \frac{288a}{b}$, so $2m_2 > m_1$. Similarly, $2n_2 > n_1$. Let $m_3 = 2m_2 - m_1$ and $n_3 = 2n_2 - n_1$. We have $\frac{m_3}{a} = \frac{2m_2}{a} - \frac{m_1}{a}$, $\frac{n_3}{b} = \frac{2n_2}{b} - \frac{n_1}{b}$, so $\frac{a}{m_3} = \frac{b}{n_3}$. Call z this last quantity. We have $m_1n_2 = \frac{a}{u} \cdot \frac{b}{v} = \frac{b}{u} \cdot \frac{a}{v} = n_1m_2$ and $m_1n_1 \cdot m_2n_2 = 200 \cdot 288 = 240^2$, so $m_1n_2 = m_2n_1 = 240$.

Finally, $m_3n_3 = 4m_2n_2 + m_1n_1 - 2m_1n_2 - 2m_2n_1 = 4 \cdot 288 + 200 - 4 \cdot 240 = 392$. But this means that dividing the sides of the rectangle into m_3 (respectively n_3), segments of length z, we get 392 squares of side length z.

Problem 3. Let p, q and r be three prime numbers such that $5 \leqslant p < q < r$. Given that $2p^2 - r^2 \geqslant 49$ and $2q^2 - r^2 \leqslant 193$, find p, q, r.

Solution. From the given relations we have $2q^2 - 193 \leqslant r^2 \leqslant 2p^2 - 49$, consequently $q^2 - p^2 \leqslant 72$. On the other hand, from $5 \leqslant p < q < r$ we obtain $r \geqslant 11$, so $2p^2 \geqslant 49 + 121 = 170$, that is $p \geqslant 11$. Since $(q-p)(q+p) \leqslant 72$ and $q - p = 2$ or $q - p \geqslant 4$, there are two possibilities:

i) $q - p = 2$ and $q + p \leqslant 36$, that is $(p, q) = (11, 13)$ or $(17, 19)$;

ii) $q - p \geqslant 4$ and $q + p \leqslant 18$, in contradiction with $p \geqslant 11$.

If $(p, q) = (11, 13)$, then $145 \leqslant r^2 \leqslant 193$, so $r = 13 = q$, which is to be rejected. If $(p, q) = (17, 19)$, then $529 \leqslant r^2 \leqslant 529$, so $r = 23$. Consequently, the required primes are $p = 17$, $q = 19$, $r = 23$.

Problem 4. Let $ABCD$ be a rectangle of center O. Assume that $AB \neq BC$. The perpendicular line at O on BD intersects the lines AB and BC at points E and F, respectively. Let M and N be the midpoints of the segments CD and AD, respectively. Prove that $FM \perp EN$.

Solution. Let P be the midpoint of BC and let Q be the point where the line DC meets the line EF. Since MP passes through the midpoints of BC and CD we deduce that $MP \parallel BD$, so $FQ \perp MP$. From $MC \perp FP$ we infer that Q is the orthocenter of triangle MPF, so $PQ \perp FM$. Triangles POQ and NOE are equal, so $QP \parallel EN$. Consequently, $FM \perp EN$.

8th GRADE

Problem 1. The lengths of the edges of a tetrahedron are natural numbers, such that the product of lengths of any pair of opposite edges is equal to 6. Show that the tetrahedron is a regular triangular pyramid with the property that the angle between a lateral edge and the plane of the base is larger than or equal to 30°.

Solution. The possible values of the side length of the tetrahedron are 1, 2, 3 or 6. If one of the edges, say AB, has length 1, then the faces containing AB have to be equilateral triangles by the triangle inequality, contradicting the hypothesis. Thus all edges have length 2 or 3: three of length 2 and three of length 3. Consequently, either $AB = AC = AD = 2$ and $BC = CD = DB = 3$ or $AB = AC = AD = 3, BC = CD = DB = 1$. In both cases the tetrahedron is a regular pyramid, the first case providing the smallest angle between a lateral edge and the base plane. Denoting this angle by u, we get $\sin u = \frac{1}{2}$ (the altitude of the pyramid is 1 and apothema is 2), so $u = 30°$.

Problem 2. A sequence of four even decimal digits, no digit of which occurs three or four times, is called *admissible*.

a) Determine the number of admissible sequences.

b) For every natural number n, $n \geqslant 2$, we denote by d_n the number of ways to complete a table with n rows and 4 columns whose entries are even decimal digits, such that the following conditions are fulfilled:

i) every row is an admissible sequence;

ii) the admissible sequence 2, 0, 0, 8 occurs on a single row of the table.

Determine the values of n such that the number $\frac{d_{n+1}}{d_n}$ is an integer.

Solution. a) Let S be the number of admissible sequences. Since the number of even digits is 5, there are $5^4 = 625$ sequences of 4 even digits. Among these 5 sequences have equal digits, and $5 \cdot 4^2 = 80$ sequences have exactly 3 equal digits. So, the required number is $S = 625 - 5 - 80 = 540$.

b) There are n ways to insert the admissible sequence 2, 0, 0, 8 as a row in the table. The remaining $n - 1$ rows can be filled in $S - 1$ ways. Consequently, $d_n = n(S-1)^{n-1} = n \cdot 539^{n-1}$, so $\frac{d_{n+1}}{d_n} = \frac{(n+1) \cdot 539}{n}$.

Thus, a necessary and sufficient condition for $\frac{d_{n+1}}{d_n}$ to be an integer is that $n \mid 539$; that is, $n \in \{7, 11, 49, 77, 539\}$.

Problem 3. Let $a, b \in [0, 1]$. Prove the inequality:

$$\frac{1}{1+a+b} \leqslant 1 - \frac{a+b}{2} + \frac{ab}{3}.$$

Solution. By brute force $2a^2b+2ab^2-3a^2-3b^2-4ab+3a+3b \geqslant 0$. Rearrange terms to get $2a(a-1)(b-1)+2b(a-1)(b-1)+a(1-a)+b(1-b) \geqslant 0$, which holds true since all terms are non-negative by the given conditions.

Problem 4. Consider the cube $ABCDA'B'C'D'$. On the edges $(A'D')$, $(A'B')$, and $(A'A)$ consider the points M_1, N_1, and P_1 respectively. On the edges (CB), (CD), and (CC') consider the points M_2, N_2, and P_2 respectively. Denote by d_1 the distance between the lines M_1N_1 and M_2N_2, by d_2 the distance between the lines N_1P_1 and N_2P_2, and by d_3 the distance between the lines P_1M_1 and P_2M_2. Suppose that the distances d_1, d_2, and d_3 are pairwise distinct. Show that the lines M_1M_2, N_1N_2, and P_1P_2 are concurrent.

Solution. Denote by a the side length of the cube. If M_1N_1 is not parallel to M_2N_2 and P_1N_1 is not parallel to P_2N_2 we get $d_1 = d_2 = a$, a contradiction. So, either $M_1N_1 \parallel M_2N_2$ or $P_1N_1 \parallel P_2N_2$. If $M_1N_1 \parallel M_2N_2$ and P_1N_1 is not parallel to P_2N_2, then $M_1P_1 \parallel M_2P_2$, for otherwise $d_2 = d_3 = a$. Thus, either $M_1N_1 \parallel M_2N_2$ and $P_1N_1 \parallel P_2N_2$ or $M_1N_1 \parallel M_2N_2$ and $M_1P_1 \parallel M_2P_2$. In either case the planes $(M_1N_1P_1)$ and $(M_2N_2P_2)$ are parallel. Consequently, $M_1N_1 \parallel M_2N_2$, $P_1N_1 \parallel P_2N_2$ and $M_1P_1 \parallel M_2P_2$. Since the lines M_1M_2, N_1N_2 and P_1P_2 are not coplanar and have non-empty mutual intersections, we conclude that they are concurrent.

9th GRADE

Problem 1. Determine the functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(x^2 + f(y)) = xf(x) + y,$$

for all $x, y \in \mathbb{N}$.

Solution. For $x = 0$ the equality reads $f(f(y)) = y$ for all $y \in \mathbb{N}$, so f is onto. Setting $x = 1$ in the given equation yields $f(1 + f(y)) = f(1) + y$ so $f(1 + x) =$ upon substitution $y = f(x)$; in particular $f(0) = 0$. By induction, $f(n) = nf(1)$, for all $n \in \mathbb{N}$. Because f is onto we must have $f(1) = 1$ (if not the image of f is the set $f(1)\mathbb{N} = \{f(1)n \mid n \in \mathbb{N}\}$), so f is the identity map.

Problem 2. a) Show that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{2n}} > n, \forall n \in \mathbb{N}^*$.

b) Prove that

$$\min\left\{k \in \mathbb{N},\ k \geqslant 2;\ \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > n\right\} > 2^n,$$

for all $n \in \mathbb{N}^*$.

Solution. a) $\frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \cdots + (\frac{1}{2^{2n-1}+1} + \frac{1}{2^{2n-1}+2} + \cdots + \frac{1}{2^{2n}}) > \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{2n-1}}{2^{2n}} = 2n \cdot \frac{1}{2} = n$, for all $n \in \mathbb{N}^*$.

b) By part a) $\{k \in \mathbb{N},\ k \geqslant 2;\ \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > n\}$ is a non-empty set of non-negative integers, for all $n \in \mathbb{N}^*$, so it has a smallest element x_n. We prove by induction that $x_n > 2^n$ for all $n \in \mathbb{N}^*$. For $n = 1$: $\frac{1}{2} + \frac{1}{3} < 1$, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$, so $x_1 = 4 > 2^1$. If $x_n > 2^n$ for some $n \geqslant 1$, then $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \leqslant n$. Since $\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}} < \frac{2^n}{2^n+1} < 1$ for $n \geqslant 2$, the two inequalities add up to yield $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^{n+1}} < n + 1$, showing that $x_{n+1} > 2^{n+1}$.

Problem 3. Consider $n \in \mathbb{N}^*$ and the real numbers a_i, $i = 1, 2, \ldots, n$, with $|a_i| \leqslant 1$ and $\sum\limits_{i=1}^{n} a_i = 0$.

Show that $\sum\limits_{i=1}^{n} |x - a_i| \leqslant n$, for all $x \in \mathbb{R}$ such that $|x| \leqslant 1$.

Solution. We may (and will) assume the following ordering and notations $a_0 = -1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant 1 = a_{n+1}$. Since $|x| \leqslant 1$, it follows that $a_k \leqslant x \leqslant a_{k+1}$ for some $k \in \{1, 2, \ldots, n\}$.

$E(x) = \sum\limits_{i=1}^{n} |x - a_i| = kx - \sum\limits_{i=1}^{k} a_i + \sum\limits_{j=k+1}^{n} a_j - (n-k)x$. For $2k \leqslant n$, use the conditions in the statement to write

$$E(x) = -nx + 2\sum_{i=1}^{k}(x - a_i) \leqslant -nx + 2k(x+1) \leqslant n.$$

Similarly, for $2k \geqslant n$:

$$E(x) = nx+2\sum_{j=k+1}^{n}(a_j-x) \leqslant -nx+2(n-k)(1-x) \leqslant nn-(2k-n)(1-x) \leqslant n.$$

Alternative proof. As a function of x on $[-1,1]$, E is continuous, piecewise linear and convex (the minimum being assumed at a_i where the sum of the a_k changes sign). Consequently, E achieves its maximum at one of the endpoints, where $E(-1) = E(1) = n$.

Problem 4. On the sides of triangle ABC consider the points $C_1, C_2 \in (AB)$, $B_1, B_2 \in (AC)$, $A_1, A_2 \in (BC)$ such that triangles $A_1B_1C_1$ and $A_2B_2C_2$ have the same centroid.

Show that the sets $[A_1B_1] \cap [A_2B_2]$, $[B_1C_1] \cap [B_2C_2]$, $[C_1A_1] \cap [C_2A_2]$ are nonempty.

Solution. Since triangles $A_1B_1C_1$ and $A_2B_2C_2$ have the same centroid $\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} + \overrightarrow{C_1C_2} = \overrightarrow{0}$.

Since $A_1, A_2 \in (BC)$, $B_1, B_2 \in (CA)$, $C_1, C_2 \in (AB)$, it follows that $\overrightarrow{A_1A_2} = \alpha\overrightarrow{BC}$, $\overrightarrow{B_1B_2} = \beta\overrightarrow{CA}$, $\overrightarrow{C_1C_2} = \gamma\overrightarrow{AB}$, for some $\alpha, \beta, \gamma \in \mathbb{R}$.

Then $\alpha\overrightarrow{BC} + \beta\overrightarrow{CA} + \gamma\overrightarrow{AB} = \overrightarrow{0}$ which is equivalent to $\alpha\overrightarrow{BC} + \beta\overrightarrow{CA} - \gamma(\overrightarrow{BC} + \overrightarrow{CA}) = \overrightarrow{0}$ or $(\alpha-\gamma)\overrightarrow{BC} = (\gamma-\beta)\overrightarrow{CA}$. Since the vectors $\overrightarrow{BC}$ and $\overrightarrow{CA}$ are linearly independent we get $\alpha = \beta = \gamma$. To make a choice, let $C_1 \in (AC_2)$ and infer that $A_1 \in (BA_2)$ and $B_1 \in (CB_2)$. The conclusion follows.

10th GRADE

Problem 1. Consider the triangle ABC and the points $D \in (BC)$, $E \in (CA)$, $F \in (AB)$, such that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}.$$

Prove that if the circumcenters of triangles DEF and ABC coincide, then the triangle ABC is equilateral.

Solution. Consider complex coordinates, the origin being taken in the circumcenter of the triangle ABC and denote with small letters the coordinates of the points. Then, if $\frac{BD}{DC} = k$, then $d = \frac{b+kc}{1+k}$ and so on.

The triangles DEF and ABC have the same circumcenter if and only if $|d| = |e| = |f|$, that is $d\bar{d} = e\bar{e} = f\bar{f}$.

Since $a\bar{a} = b\bar{b} = c\bar{c}$, this amounts to $a\bar{b} + b\bar{a} = a\bar{c} + c\bar{a} = b\bar{c} + c\bar{b}$, which is equivalent with $|a-b|^2 = |a-c|^2 = |b-c|^2$, whence the conclusion.

Problem 2. Let a, b, c be three complex numbers such that

$$a|bc| + b|ca| + c|ab| = 0.$$

Prove that

$$|(a-b)(b-c)(c-a)| \geqslant 3\sqrt{3}|abc|.$$

Solution. If one of the numbers is nil, then the conclusion is obvious.

Otherwise, dividing by $|abc|$ and denoting $\alpha = \frac{a}{|a|}$, $\beta = \frac{b}{|b|}$, $\gamma = \frac{c}{|c|}$, the hypothesis becomes $\alpha + \beta + \gamma = 0$ and $|\alpha| = |\beta| = |\gamma| = 1$. It is a well-known fact that, in this case, the differences between the arguments of the numbers α, β, γ are $\pm\frac{2\pi}{3}$.

The Cosine Theorem gives now $|a-b|^2 = |a|^2 + |b|^2 + |a|\,|b| \geqslant 3|a|\,|b|$ and two other similar relations. Multiplication of the tree inequalities yields the desired result.

Problem 3. Consider the set $A = \{1, 2, 3, \ldots, 2008\}$. We say that a set is *of type* r, $r \in \{0, 1, 2\}$, if that set is a nonempty subset of A and the sum of its elements gives the remainder r when divided by 3. Denote by X_r, $r \in \{0, 1, 2\}$ the class of sets of type r.

Determine which of the classes X_r, $r \in \{0, 1, 2\}$, is the largest.

Solution. Add to X_0 the empty set and denote $X_{r,n}$ the class of the subsets of type r of the set $\{1, 2, \ldots, n\}$. If $n \in \mathbb{N}^*$, then $n+1, n+2, n+3$, are three numbers a, b, c such that $a \equiv 0 \bmod 3$, $b \equiv 1 \bmod 3$, $c \equiv 2 \bmod 3$. We notice that $X_{0,n+3}$ is made of:

– the sets from $X_{0,n}$;

– the sets in $X_{0,n}$ along with $\{a\}$, $\{b,c\}$ or $\{a,b,c\}$;
– the sets in $X_{1,n}$ along with $\{c\}$ or $\{a,c\}$;
– the sets in $X_{2,n}$ along with $\{b\}$ or $\{a,b\}$.
Argue similarly for $X_{1,n+3}$ and $X_{2,n+3}$ to get

$$|X_{0,n+3}| = 4|X_{0,n}| + 2|X_{1,n}| + 2|X_{2,n}|,$$
$$|X_{1,n+3}| = 2|X_{0,n}| + 4|X_{1,n}| + 2|X_{2,n}|,$$
$$|X_{2,n+3}| = 2|X_{0,n}| + 2|X_{1,n}| + 4|X_{2,n}|.$$

Since $|X_{0,1}| = |X_{1,1}| = 1$ and $|X_{2,1}| = 0$, an obvious induction leads now to $|X_{0,3n+1}| = |X_{1,3n+1}| > |X_{2,3n+1}|$. Then, from $2008 = 3 \cdot 669 + 1$ and $|X_0| = |X_{0,2008}| - 1$ follows $|X_1| > |X_0| > |X_2|$.

Problem 4. Consider the statement $p(n) : (n^2+1)|n!$, $n \in \mathbb{N}$. Show that the sets

$$A = \{n \in \mathbb{N} \mid p(n) \text{ is true}\} \quad \text{and} \quad F = \{n \in \mathbb{N} \mid p(n) \text{ is false}\}$$

are infinite.

Solution. We notice first that a good chance of getting elements of A is to try taking $n = 2m^2$. Indeed, in this case $n^2 + 1 = 4m^4 + 1 = (2m^2+1)^2 - 4m^2 = (2m^2-2m+1)(2m^2+2m+1)$, the factors $(2m^2-2m+1)$ and $(2m^2+2m+1)$ are coprime and $2m^2 - 2m + 1 < 2m^2$.

Then, for $m = 5p+1$, $2m^2+2m+1 = 5(10p^2+6p+1)$ and, for $p \neq 0$, the factors 5, $10p^2+6p+1$ are coprime and less than n. Hence, $(n^2+1)|n!$ for each n of the form $2(5p+1)^2$.

An element of F can be obtained if we find a prime number p and $n \in \mathbb{N}$ such that $p \mid n^2+1$ and $p > n$. This goal can be attained if we take a prime p having a multiple of the form m^2+1 and take n as the remainder of $m \pmod p$.

We prove by contradiction that the set of primes of this type is infinite. Indeed, if the set of these primes were $P = \{p_1, \dots, p_n\}$, then the prime factors of the number $(p_1p_2\dots p_n)^2 + 1$ would not be elements of P, contradiction.

11th GRADE

Problem 1. Let $f : (0,\infty) \to \mathbb{R}$ be a continuous function, such that for any $x \in (0,\infty)$ the sequence $(f(nx))_{n\in\mathbb{N}^*}$ is nondecreasing.

Prove that f is a nondecreasing function.

Solution. Let $x, y \in \mathbb{Q}_+^*$, $x < y$. Let $m, n, p \in \mathbb{N}^*$, $m < n$, such that $x = \frac{m}{p}$ and $y = \frac{n}{p}$. By the hypothesis, the sequence $(f(k \cdot \frac{1}{p}))_{k\in\mathbb{N}^*}$ is nondecreasing, so $f(x) = f(m \cdot \frac{1}{p}) \leqslant f(n \cdot \frac{1}{p}) = f(y)$, i.e. f is nondecreasing on $\mathbb{Q}_+^*$.

Next, let $x, y \in \mathbb{R}_+^*$, $x < y$, and consider two sequences of positive rational numbers, $(r_n)_{n\in\mathbb{N}^*}$ and $(r'_n)_{n\in\mathbb{N}^*}$, such that $x < r_n < r'_n < y$ for every $n \in \mathbb{N}^*$, and $\lim_{n\to\infty} r_n = x$, $\lim_{n\to\infty} r'_n = y$.

By monotonicity of f on $\mathbb{Q}_+^*$ and continuity of f,

$$f(x) = \lim_{n\to\infty} f(r_n) \leqslant \lim_{n\to\infty} f(r'_n) = f(y),$$

which shows f is nondecreasing on $\mathbb{R}_+^*$.

Problem 2. Prove that an invertible matrix $A \in \mathcal{M}_n(\mathbb{C})$ has the property $A^{-1} = \overline{A}$ if and only if there exists an invertible matrix $B \in \mathcal{M}_n(\mathbb{C})$ such that $A = B^{-1} \cdot \overline{B}$.

Solution. Assume first that $A = B^{-1} \cdot \overline{B}$ for some B. Then

$$A \cdot \overline{A} = B^{-1} \cdot \overline{B} \cdot \overline{(B^{-1})} \cdot B = B^{-1} \cdot \overline{B}(\overline{B})^{-1} \cdot B = I_n,$$

so $A^{-1} = \overline{A}$.

For the converse, suppose that $A^{-1} = \overline{A}$ and consider a matrix B of the form $\alpha \cdot \overline{A} + \beta \cdot I_n$ for $\alpha, \beta \in \mathbb{C}$. We shall prove that α, β can be chosen so that B be invertible and $A = B^{-1} \cdot \overline{B}$. We have

$$A = B^{-1} \cdot \overline{B} \Leftrightarrow B \cdot A = \overline{B} \Leftrightarrow (\alpha \cdot \overline{A} + \beta \cdot I_n) \cdot A = \overline{\alpha} \cdot A + \overline{\beta} \cdot I_n$$
$$\Leftrightarrow \alpha \cdot \overline{A} \cdot A + \beta \cdot A = \overline{\alpha} \cdot A + \overline{\beta} \cdot I_n \Leftrightarrow \alpha \cdot I_n + \beta \cdot A = \overline{\alpha} \cdot A + \overline{\beta} \cdot I_n.$$

If we set $\beta = \overline{\alpha}$ the last equality is trivially satisfied, so we only need to make sure that $B = \alpha\overline{A} + \overline{\alpha} I_n$ is invertible for some $\alpha \in \mathbb{C}$.

For $\alpha \neq 0$ we have

$$\det B = \det(\alpha\overline{A} + \overline{\alpha}I_n) = \alpha^n \det\left(\overline{A} + \frac{\overline{\alpha}}{\alpha}I_n\right).$$

Since the image of the map $\alpha \mapsto \frac{\overline{\alpha}}{\alpha}$ ($\alpha \in \mathbb{C}^*$) is the unit circle (an infinite set), and since the zeros of $\det(\overline{A} + zI_n)$ are the negatives of the eigenvalues of $\overline{A}$ (hence finitely many), some $\alpha \in \mathbb{C}^*$ yields an invertible B, as desired.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on $\mathbb{R}$ such that there exists $c \in \mathbb{R}$ with

$$\frac{f(b) - f(a)}{b - a} \neq f'(c), \quad \text{for all } a, b \in \mathbb{R},\ a \neq b.$$

Prove that $f''(c) = 0$.

Solution. Since $\frac{f(b)-f(a)}{b-a} \neq f'(c)$ for every $a, b \in \mathbb{R}$, $a \neq b$, it follows that the map $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(x) - x \cdot f'(c)$ is injective, hence strictly monotonic. Therefore, $g'(x) = f'(x) - f'(c)$ has constant sign, and we see that c must be an extremal point of f'. This shows that $f''(c) = 0$, and concludes the proof.

Alternative solution (Francisc Bozgan and Gabriel Dospinescu). We first prove the following

LEMMA. Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a differentiable function. Consider the set

$$J = \left\{ \frac{f(x) - f(y)}{x - y} : x, y \in I, x \neq y \right\}.$$

Then J is an interval and a dense subset of $f'(I)$ (therefore, $|f'(I) \setminus J| \leqslant 2$).

Proof. By the mean value theorem, it follows that J is a subset of $f'(I)$. It is clear that the closure of J contains $f'(I)$, so J is dense in $f'(I)$. It remains to prove that J is an interval. Since $f'(I)$ is also an interval (f' has the intermediate value property) it will follow that $|f'(I) \setminus J| \leqslant 2$.

Consider the set $C = \{(x, y) \in I \times I : x < y\}$ and define a map $g : C \to \mathbb{R}$ by $g(x, y) = \frac{f(x)-f(y)}{x-y}$. Since C is connected and g is continuous, we get that $J = g(C)$ is also connected, so J is an interval, as desired.

Back to our problem, let $I = \mathbb{R}$ let J be as in the lemma. Since $f'(c) \notin J$, the lemma implies that $f'(c)$ is one of the endpoints of $f'(I)$, i.e. an extremal point of f'. Conclude as in the first solution.

Problem 4. Let $A \in \mathcal{M}_n(\mathbb{R})$ be an antisymmetric matrix ($\forall i, j, a_{ij} + a_{ji} = 0$). Prove that

$$\det(A + xI_n) \cdot \det(A + yI_n) \geqslant \det(A + \sqrt{xy}I_n)^2,$$

for all $x, y \in [0, \infty)$.

Solution. Let $P(x) = \det(A + xI_n) = x^n + a_{n-1} + \cdots + a_1x + \det A$. Each a_k is the sum of all principal minors of order k of the matrix A. Since all principal minors are nonnegative, being the determinants of some skew-symmetric matrices, it follows that all $a_k \geqslant 0$ for every $k = \overline{1, n}$.

By the Cauchy–Schwarz inequality, $P(x) \cdot P(y) \geqslant \left(P(\sqrt{xy})\right)^2$ for all $x, y \geqslant 0$.

Alternative solution (Given by several contestants). Since A is skew-symmetric, its eigenvalues are purely imaginary. It follows that the nonzero roots of $P(X) = \det(A + XI_n)$ come in pairs $\{z_j, \overline{z_j}(= -z_j)\}$, $j = \overline{1, k}$, and $r = n - 2k$ is the multiplicity of 0 in P, so

$$P(X) = X^r \cdot \prod_{j=1}^{k} (X^2 + |z_j|^2).$$

Since $(x^2 + |z_j|^2) \cdot (y^2 + |z_j|^2) \geqslant (xy + |z_j|^2)^2$ for all $x, y \in \mathbb{R}$ and all $j = \overline{1, k}$, the conclusion follows.

REMARKS. 1) The determinant of any skew-symmetric matrix can be written as the square of a homogeneous polynomial in the entries, called the *Pfaffian* of the matrix. For odd dimensional skew-symmetric matrices, the Pfaffian is always zero.

2) The statement that the eigenvalues of a skew-symmetric matrix being purely imaginary is equivalent to the statement that all eigenvalues of a Hermitian matrix are real (a matrix with complex entries is called *Hermitian* if it is equal to its own conjugate transpose). The equivalence is a straightward consequence of the fact that a matrix A is skew-symmetric if and only if iA is Hermitian.

To prove the statement about the eigenvalues of a Hermitian matrix, consider one such matrix $A = A^*$ (for every matrix X with complex entries we denote by X^* its conjugate transpose). Let λ be an eigenvalue of A and let x be an eigenvector corresponding to λ.

Using the fact that * is an involution ($(X^*)^* = X$), and the identity $(XY)^* = Y^*X^*$, we get

$$(x^*Ax)^* = x^*A^*x = x^*Ax.$$

Since $x^*Ax = \lambda \cdot x^*x$, it follows that $\lambda \cdot x^*x = \overline{\lambda} \cdot x^*x$, but x^*x is nonzero, hence λ is real.

12th GRADE

Problem 1. Let a be a positive real number and let $f : [0, \infty) \to [0, a]$ be a function which has the intermediate value property on $[0, \infty)$ and is continuous on $(0, \infty)$. If $f(0) = 0$ and

$$xf(x) \geqslant \int_0^x f(t)\mathrm{d}t, \quad \text{for all } x \in (0, \infty),$$

prove that f has antiderivatives on $[0, \infty)$.

Solution. Since f is continuous on $(0, \infty)$ and bounded, it follows that f is integrable on $[0, x]$, for every $x \geqslant 0$. The function $F : [0, \infty) \to \mathbb{R}$, defined by

$$F(x) = \int_0^x f(t)\mathrm{d}t,$$

is therefore differentiable on $(0, \infty)$, and $F'(x) = f(x)$ for all $x > 0$. Define a map $g : (0, \infty) \to \mathbb{R}$ by letting $g(x) = F(x)/x$ for $x > 0$. Since

$$g'(x) = \frac{xf(x) - F(x)}{x^2} \geqslant 0,$$

for all $x > 0$, g is nondecreasing, so $\lim\limits_{x\to 0} g(x)$ exists. Let ℓ denote this limit.

Since f has the intermediate value property, there exists a sequence (a_n) of positive real numbers such that $a_n \to 0$ and $f(a_n) \to f(0) = 0$. By hypothesis, $f(a_n) \geqslant g(a_n) \geqslant 0$, whence $\lim\limits_{n\to\infty} g(a_n) = 0$, so $\ell = 0$. Since $F(0) = 0$ and $g(x) = F(x)/x$, we get

$$F'(0) = \lim_{x\to 0} \frac{F(x) - F(0)}{x} = \lim_{x\to 0} g(x) = \ell = 0,$$

so $F'(0) = f(0)$ and F is an antiderivative of f on $[0, \infty)$.

Problem 2. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function, whose derivative f' is continuous on $[0, 1]$. Prove that if $f(1/2) = 0$, then

$$\int_0^1 (f'(x))^2\,\mathrm{d}x \geqslant 12 \left(\int_0^1 f(x)\mathrm{d}x\right)^2.$$

Solution. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\left(\int_0^{1/2} xf'(x)\mathrm{d}x\right)^2 &\leqslant \left(\int_0^{1/2} x^2\mathrm{d}x\right)\left(\int_0^{1/2} (f'(x))^2\mathrm{d}x\right)\\ &= \frac{1}{24}\int_0^{1/2} (f'(x))^2\mathrm{d}x,\\ \left(\int_{1/2}^1 (1-x)f'(x)\mathrm{d}x\right)^2 &\leqslant \left(\int_{1/2}^1 (1-x)^2\mathrm{d}x\right)\left(\int_{1/2}^1 (f'(x))^2\mathrm{d}x\right)\\ &= \frac{1}{24}\int_{1/2}^1 (f'(x))^2\mathrm{d}x,\end{aligned}$$

so

$$\frac{1}{24}\int_0^1 (f'(x))^2\mathrm{d}x \geqslant \left(\int_0^{1/2} xf'(x)\mathrm{d}x\right)^2 + \left(\int_{1/2}^1 (1-x)f'(x)\mathrm{d}x\right)^2.$$

Integration by parts along with $f(1/2) = 0$ yields

$$\int_0^{1/2} xf'(x)\mathrm{d}x = -\int_0^{1/2} f(x)\mathrm{d}x \quad \text{and} \quad \int_{1/2}^1 (1-x)f'(x)\mathrm{d}x = \int_{1/2}^1 f(x)\mathrm{d}x.$$

Consequently,

$$\frac{1}{24}\left(\int_0^1 (f'(x))^2 \mathrm{d}x\right) \geqslant \left(\int_0^{1/2} f(x)\mathrm{d}x\right)^2 + \left(\int_{1/2}^1 f(x)\mathrm{d}x\right)^2$$
$$\geqslant \frac{1}{2}\left(\int_0^{1/2} f(x)\mathrm{d}x + \int_{1/2}^1 f(x)\mathrm{d}x\right)^2$$
$$= \frac{1}{2}\left(\int_0^1 f(x)\mathrm{d}x\right)^2.$$

The conclusion follows.

Problem 3. Let A be a finite unitary ring with n elements, such that the equation $x^n = 1$ has the unique solution $x = 1$ in A. Prove that:

(a) 0 is the unique nilpotent element of the ring A;

(b) there exists $k \in \mathbb{N}$, $k \geqslant 2$, such that the equation $x^k = x$ has n solutions in A.

($x \in A$ is *nilpotent* if there exists $m \in \mathbb{N}^*$ such that $x^m = 0$.)

Solution. (a) Suppose there is a nonzero nilpotent $x \in A$ and let m be its order of nilpotency (the smallest positive integer for which $x^m = 0$). Then $y = x^{m-1}$ has order of nilpotency 2, and we have $(1-y)^n = 1 - ny = 1$. The hypothesis forces $1 - y$ to be equal to 1, that is $y = 0$, a contradiction.

(b) Let $x_1, x_2, \ldots, x_n$ denote the elements of A. The set

$$\{(x_1^i, x_2^i, \ldots, x_n^i) : i \in \mathbb{N}^*\}$$

is a subset of A^n, hence finite. We can thus find $1 \leqslant l < m$ such that $x^l = x^m$ for all $x \in A$. For each $x \in A$, $x(x^{m-l} - 1)$ is then nilpotent, and therefore zero by part (a). If we take $k = m - l + 1$ we have $k \geqslant 2$ and $x^k = x$ for every $x \in A$, as desired.

REMARKS. 1) Since $x^k = x$ for all $x \in A$, a celebrated commutativity result of Jacobson implies the commutativity of A. We shall prove that A is a product of fields. If A contains a nontrivial idempotent e, $A \simeq B \times C$, where $B = Ae$, $C = A(1-e)$. Clearly, the elements of B and C satisfy the equation $X^k = X$, so we can argue by induction that B and C are products of fields. It follows that A is

also a product of fields. If A contains no nontrivial idempotents, for each $x \in A$, x^{k-1} is idempotent and therefore equals 0 or 1. Consequently, every $x \in A$ is either zero or an invertible element, so A is a field

2) The same conclusion can be drawn by the Wedderburn-Artin structure theorem for semisimple rings: A is artinian and Jacobson semisimple (its Jacobson radical is zero by part (a) since it consists entirely of nilpotents), therefore it is a semisimple ring, and by the structure theorem we can write it as a product of matrix algebras over division rings. Since A contains no nilpotents, and finite division rings are commutative, we conclude that it must be a product of fields.

Problem 4. Let $\mathcal{G}$ be the set of finite groups with at least two elements.

(a) Show that if $G \in \mathcal{G}$, then

$$|\mathrm{End}(G)| \leqslant \sqrt[p]{n^n},$$

where $|\mathrm{End}(G)|$ is the number of endomorphisms of G, $n = n(G)$ is the number of elements of G, and $p = p(G)$ is the greatest prime divisor n.

(b) Determine the groups in $\mathcal{G}$ such that the inequality in (a) holds with equality.

Solution. (a) Let a be an element of G of order p, let $H = \langle a \rangle = \{e, a, \ldots, a^{p-1}\}$ and let $I = \{x_1, x_2, \ldots, x_k\}$ be a complete set of representatives for the left cosets of G modulo H, with $x_1 = a$. Clearly, $k = n/p$. Every endomorphism of G is uniquely determined by its values on I: if $x \in x_sH$, $x = x_s a^t$ for some t, whence $f(x) = f(x_s)f(a)^t = f(x_s)f(x_1)^t$. Consequently, $|\mathrm{End}(G)| \leqslant |G^I| = n^{n/p} = \sqrt[p]{n^n}$.

(b) Equality holds in (a) iff $|\mathrm{End}(G)| = |G^I|$, that is iff any map $f : I \to G$ extends to an endomorphism of G. Suppose that G is such that the previous equivalent statements hold. If $k = 1$, $|G| = p$, so $G \cong (\mathbb{Z}_p, +)$ and $|\mathrm{End}(\mathbb{Z}_p, +)| = p = \sqrt[p]{p^p}$. Assume now $k \geqslant 2$. Let x be any element of G and consider the map $f : I \to G$ given by $f(x_1) = x$, $f(x_i) = e$ for $i > 1$. The map f extends to an endomorphism of G by assumption, so

$$x^p = f(x_1^p) = f(e) = e.$$

If $x_2^{-1} \notin x_2H$, we can find a map $f : I \to G$, sending x_2 to e, and x_2^{-1} to some other element of G. But such an f does not extend to an endomorphism of

G, contradicting the assumption. Therefore $x_2^{-1} \in x_2H$, or equivalently $x_2^2 \in H$. If p is odd, it follows that $1-p$ is even, so $x_2 = x_2^{1-p} \in H$, again a contradiction. Hence we must have $p = 2$ and G commutative ($x^2 = e$ for all $x \in G$). If $k = 2$, $G = K$, the Klein four-group, which is easily seen to satisfy the desired equality. If $k \geqslant 3$, x_2x_3 does not belong to any of the cosets x_iH, $i = 2, 3$, so we can find a map $f : I \to G$, with $f(x_2) = f(x_3) = e$ and $f(x_2x_3) \neq e$. This is again contradictory, since we cannot extend f to an endomorphism of G.

Summing up, the only groups for which equality holds in (a) are $\mathbb{Z}_p$, with p prime, and the Klein four-group.

PROBLEMS AND SOLUTIONS

BMO AND IMO SELECTION TESTS

Problem 1. Determine all families $\mathcal{F}$ of $n \geqslant 1$ integers such that no sum of elements of a non-empty subfamily of $\mathcal{F}$ is divisible by $n+1$.

(How many such families exist, made of distinct positive integers between 1 and n^2+n?)

Solution. Denote by $a_1, a_2, \ldots, a_n$ the elements of such a family. Consider the sums $s_k = \sum_{i=1}^{k} a_i$, $k = 1, 2, \ldots, n$. None should be divisible by $n+1$, and no two should be congruent modulo $n+1$ (otherwise their difference would be a sum divisible by $n+1$), hence they must represent all n nonzero residues modulo $n+1$.[1] But so are then the sums $\sigma_1 = a_2$, $\sigma_2 = a_2 + a_1$, $\sigma_k = s_k = \sum_{i=1}^{k} a_i$, $k = 3, \ldots, n$, hence we need have $s_1 = a_1$ and $\sigma_1 = a_2$ congruent modulo $n+1$.

Since the indexing has been done arbitrarily, it follows that any two elements of the family must be congruent to the same value a. Now, any subfamily with $1 \leqslant k \leqslant n$ elements will have as sum of its elements a value congruent to ka, and we need this to be a nonzero residue, hence a and $n+1$ must be co-prime.

Therefore, such a family must be of the form

$$\mathcal{F} = \{a + k_i(n+1);\ k_i \in \mathbb{Z},\ 1 \leqslant i \leqslant n,\ a \in \mathbb{Z},\ (a, n+1) = 1\},$$

and this describes all solutions.

(The number of families made of distinct positive integers between 1 and n^2+n is therefore equal to $\varphi(n+1)$, since $1 + n(n+1) = n^2 + n + 1$ is too large (while $n + (n-1)(n+1) = n^2 + n - 1 < n^2 + n$ is OK).)

[1] This is of course reminiscent of Erdős' early and folklore method in showing that such a family must have a partial sum divisible by n.

Problem 2. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers so that $a_i < b_i$, for all $i = 1, 2, \ldots, n$, and $b_1 + b_2 + \cdots + b_n < 1 + a_1 + a_2 + \cdots + a_n$. Prove there exists $c \in \mathbb{R}$ such that for all $i = 1, 2, \ldots, n$ and $k \in \mathbb{Z}$

$$(a_i + k + c)(b_i + k + c) > 0.$$

Solution. The condition is equivalent to no integer $-k$ to be found in any interval $[a_i + c, b_i + c]$, i.e. in the translates by c of $[a_i, b_i]$. Moreover, the sum of the lengths of these intervals is known to be less than 1, since $\sum_{i=1}^{n}(b_i - a_i) < 1$.

We are thus asked to show that the union S of a (finite) set of non-degenerate intervals, of total length less than 1, may be translated such that it becomes disjunct of $\mathbb{Z}$. For all $k \in \mathbb{Z}$, consider the sets $S_k = S \cap [k, k+1)$ and their translates $S_k - k \subset [0, 1)$. Due to possible overlaps, the total length of the union $\cup\{S_k - k; k \in \mathbb{Z}\}$ is at most equal to the total length of the union $\cup\{S_k; k \in \mathbb{Z}\} = S$, hence less than 1. Therefore, there exists $-c \in [0,1) \setminus \cup\{S_k - k; k \in \mathbb{Z}\}$, whence $(S + c) \cap \mathbb{Z} = \emptyset$.[2]

Problem 3. A convex hexagon $ABCDEF$ has all sides of length 1. Prove that one of the radii of the circumscribed circles of the triangles ACE and BDF is at least 1 long.

Solution. Suppose both radii are shorter than 1. Let O be the circumcenter of triangle ACE. Then $\angle AOC > \angle ABC$, $\angle COE > \angle CDE$, $\angle EOA > \angle EFA$. The following possibilities may occur:

a) triangle ACE is not obtuse-angled, hence it contains O, therefore $\angle AOC + \angle COE + \angle EOA = 2\pi$;

b) triangle ACE is obtuse-angled,[3] hence it does not contain O, therefore $\angle AOC + \angle COE + \angle EOA < 2\pi$.

[2] This is the one-dimensional equivalent of the fact that a family of simple closed curves of total interior area less than 1 may be positioned in the plane such that it contains no lattice point, with similar proof. Generalization to higher dimensions is readily done.

[3] This case, in fact, can never occur, as a little bit of trigonometry will show, like in (C. Platon) alternative algebraic solution. Then, of course, one of the radii is at most 1 long, with similar proof. Moreover, if one of the radii is of length 1, the other one is also with necessity of length 1, and this happens iff the hexagon has parallel opposite sides.

In both cases $\angle ABC + \angle CDE + \angle EFA < 2\pi$. A similar reasoning, for triangle BDF, yields $\angle BCD + \angle DEF + \angle FAB < 2\pi$. Summing these two inequalities gives the sum of the angles of the hexagon to be (strictly) less than 4π, contradiction.

Problem 4. Prove that, given any convex polygon P with n sides, there exists a set S of $n-2$ points interior to P, such that the interior of any triangle determined by three of the vertices of P contains exactly one point from S.

Solution. Let $0, 1, \ldots, n-1$ be a cyclical indexing for the vertices of P, and $*$ a fixed point interior to the side $[n-1, 0]$. For each $i \in \{1, \ldots, n-2\}$, consider one point each i' interior to the segment intercepted by the triangle $[i-1, i, i+1]$ on the segment $[i, *]$. The set of the points i' satisfies the stated requirement, since from the convexity of P, for any i, j, k distinct vertices of P, one only of the segments $[i, *]$, $[j, *]$, $[k, *]$ contains a point interior to the triangle $[i, j, k]$, while all points ℓ' with $\ell \neq 0, i, j, k, n-1$ are clearly exterior to it.

Alternative solution. This solution allows for choosing the point i' anywhere in the interior of the intersection of the pair of triangles $[i-1, i, i+1]$ and $[n-1, i, 0]$, and provides a neat proof by induction that the stated requirement is fulfilled.

Alternative solution. This solution allows for choosing the point i' anywhere in the interior of the intersection of the pair of triangles $[i-1, i, i+1]$ and $[n-i, i, n-i-1]$, for $i \neq 0$ and $i \neq \lfloor n/2 \rfloor$, but the proof is slightly more arduous.

Problem 5. Determine the greatest common divisor of the numbers

$$2^{561} - 2, 3^{561} - 3, \ldots, 561^{561} - 561.$$

Solution. More generally, for integer $n \geqslant 3$,[4] we must find the g.c.d. of the

[4] The use of 561 is a harmless joke; it hints to the notion of *Carmichael number* (odd composite n such that $a^{n-1} \equiv 1 \pmod n$ for all integers a with $(a, n) = 1$), with the smallest example being $561 = 3 \cdot 11 \cdot 17$.

In effect our solution proves Korselt's result of 1899 that n is Carmichael if and only if n is squarefree and such that $p - 1 | n - 1$ for each prime p that divides n.

It is known, since the 1990's (Alford, Granville & Pomerance), that there exist infinitely many Carmichael numbers.

numbers

$$2(2^{n-1}-1), 3(3^{n-1}-1), \ldots, n(n^{n-1}-1).$$

Let p be a prime dividing all these numbers.

If $p > n$, then $p|1^{n-1}-1, p|2^{n-1}-1, \ldots, p|n^{n-1}-1$, hence $\hat{1}, \hat{2}, \ldots, \hat{n}$ are roots for the polynomial $x^{n-1}-1$ in $\mathbb{Z}_p[x]$, therefore $x^{n-1}-1$ has n roots in $\mathbb{Z}_p$, which should imply this is the null polynomial, absurd.

If $p \leqslant n$, then $p|1^n-1, p|2^n-2, \ldots, p|n^n-n$; it follows that $p|a^{n-1}-1$, where $a \in \{1, 2, \ldots, p-1\}$, hence $\hat{1}, \hat{2}, \ldots, \widehat{p-1}$ are roots for the polynomial $x^{n-1}-1$ in $\mathbb{Z}_p[x]$. But they are also the roots of the polynomial $x^{p-1}-1$, from Fermat's Theorem. Therefore $x^{p-1}-1$ divides $x^{n-1}-1$, whence $p-1|n-1$, since for $n-1 = q(p-1)+r$, with $0 \leqslant r < p-1$, we have $x^{n-1}-1 = x^r(x^{q(p-1)}-1)+(x^r-1)$, so $x^{p-1}-1$ divides x^r-1, whence $r=0$.

Conversely, for $p-1|n-1$, it follows from Fermat's Theorem that $p|a^n-a$ for all a, hence p divides all the numbers in the statement.

Finally, since p^2 does not divide $p(p^{n-1}-1)$, p^2 cannot divide all numbers $2^n-2, 3^n-3, \ldots, n^n-n$, hence the sought after g.c.d. is squarefree.

Based on the above, the sought after g.c.d. is

$$\prod_{\substack{p \text{ prime} \\ p-1|n-1}} p.$$

In particular, the greatest common divisor of the numbers in the problem ($n = 561$, hence $n-1 = 560 = 2^4 \cdot 5 \cdot 7$) is $2 \cdot 3 \cdot 5 \cdot 11 \cdot 17 \cdot 29 \cdot 41 \cdot 71 \cdot 113 \cdot 281$.

Problem 6. Let $n \geqslant 3$ be an odd integer. Determine the maximum value of the cyclic sum

$$E = \sqrt{|x_1-x_2|} + \sqrt{|x_2-x_3|} + \cdots + \sqrt{|x_{n-1}-x_n|} + \sqrt{|x_n-x_1|},$$

for $0 \leqslant x_i \leqslant 1, i = 1, 2, \ldots, n$.

Solution. We will show $\sup E = n-2+\sqrt{2}$, the bound being attained by taking, for example, $x_1 = \frac{1}{2}, x_2 = 1, x_3 = 0, \ldots, x_{n-1} = 1, x_n = 0$.

Suppose the above maximum is achieved [5] for the values $a_1, a_2, \ldots, a_n$. Then either all a_i are 0 or 1, or there exists an a_i with $0 < a_i < 1$.

If $a_j = a_{j+1}$ for some j then $E \leqslant n-1 < n-2+\sqrt{2}$, worse than the claimed result. In the first case, $a_j = a_{j+1}$ for some j (since n odd), hence the above applies.

In the second case, $a_{i-1} < a_i$ and $a_{i+1} < a_i$ is impossible, since taking $a_i = 1$ would achieve a larger value. Similarly, $a_{i-1} > a_i$ and $a_{i+1} > a_i$ is impossible, since taking $a_i = 0$ would achieve a larger value.

Thus $a_{i-1} > a_i > a_{i+1}$ or vice-versa. In both cases

$$\sqrt{|a_{i-1}-a_i|} + \sqrt{|a_i-a_{i+1}|} \leqslant 2\sqrt{\frac{1}{2}|a_{i-1}-a_{i+1}|} \leqslant \sqrt{2}.$$

Since clearly

$$\sum_{j \neq i-1, i} \sqrt{|a_j-a_{j+1}|} \leqslant n-2,$$

the claimed result follows.

Problem 7. Does it exist a sequence of integers $1 \leqslant a_1 < a_2 < a_3 < \cdots$ such that, for any integer n, the set $\{a_k+n; k = 1, 2, 3, \ldots\}$ contains a finite number of primes?

Solution. Consider the sequence given by $a_k = ((2k)!)! + k!$.

For $|n| \geqslant 2$ and $k \geqslant |n|$, we have $n|a_k+n$ and $a_k+n > 2|n|$, so $|n|$ is a proper divisor of a_k+n.

For $n = 0$ and $k \geqslant 2$, we have $k|a_k$ and $a_k > k$, so k is a proper divisor of a_k+n.

For $n = 1$ and $k \geqslant 1$, we have $k!+1|a_k+1$ and $a_k+1 > k!+1$, so $k!+1$ is a proper divisor of a_k+n.

For $n = -1$ and $k \geqslant 3$, we have $k!-1|a_k-1$ and $a_k-1 > k!-1$, so $k!-1$ is a proper divisor of a_k+n.

Alternative solution. Consider $a_k = (k!)^3$.

[5] The fact that the supremum is achieved is a consequence of Weierstrass' Theorem, E being continuous in variables $x_1, x_2, \ldots, x_n$, defined in the compact unit n-hypercube.

The situation is clear for $n = 0$, while for $|n| \geqslant 2$ and $k \geqslant |n|$, $|n|$ is a proper divisor of $a_k + n$.

For $n = 1$ and $k \geqslant 1$, we have $k! + 1 | a_k + 1$ and $a_k + 1 > k! + 1$, so $k! + 1$ is a proper divisor of $a_k + n$.

For $n = -1$ and $k \geqslant 3$, we have $k! - 1 | a_k - 1$ and $a_k - 1 > k! - 1$, so $k! - 1$ is a proper divisor of $a_k + n$.

Problem 8. Prove that any convex pentagon has a vertex whose distance to the support line of its opposite side is strictly less than the sum of the distances from its neighbouring vertices to the same line.

Solution. Let $0, 1, 2, 3, 4$ be a cyclical indexing for the vertices of the pentagon. From all peripheral triangles $[i-1, i, i+1]$, consider one of minimal area; wlog we may assume it is $[4, 0, 1]$. We will show vertex i is one suitable for the statement. Denote by $\mathcal{A}[F]$ the area of a figure F.

The condition in the statement comes to

$$\mathcal{A}[0,2,3] < \mathcal{A}[1,2,3] + \mathcal{A}[2,3,4],$$

equivalent to

$$\begin{array}{rcl} \mathcal{A}[0,1,2,3,4] & = & \mathcal{A}[0,1,2] + \quad \mathcal{A}[0,2,3] \quad + \mathcal{A}[0,3,4] \\ & < & \mathcal{A}[0,1,2] + \mathcal{A}[1,2,3] + \mathcal{A}[2,3,4] + \mathcal{A}[0,3,4]. \end{array}$$

Let $*$ be the meeting point of diagonals $[1,3]$ and $[2,4]$. Since triangles $[1,2,3]$ and $[2,3,4]$ partially overlap, we have

$$\mathcal{A}[0,1,2,3,4] < \mathcal{A}[*,0,1] + \mathcal{A}[*,4,0] + \mathcal{A}[1,2,3] + \mathcal{A}[2,3,4].$$

Let us notice that $\mathcal{A}[*,0,1]$ is a convex combination of $\mathcal{A}[0,1,2]$ and $\mathcal{A}[4,0,1]$.

Due to the minimality of $\mathcal{A}[4,0,1]$, we have $\mathcal{A}[0,1,2] \geqslant \mathcal{A}[4,0,1]$, hence $\mathcal{A}[*,0,1] \leqslant \mathcal{A}[0,1,2]$. Similarly, $\mathcal{A}[*,4,0] \geqslant \mathcal{A}[C,3,4]$. Together, these few last inequalities yield the one we identified as needed.

Problem 9. Determine the minimum number of edges that a connected graph with $n \geqslant 3$ vertices may have, if each edge belongs to at least one triangle.

Solution. Let us first build some models having the stated properties.[6]

For odd $n = 2k+1$, a possible model is made by k triangles articulated around a common vertex (call this configuration a *k-windmill* with k wings); the number of its edges is $e = 3k = \frac{3}{2}(n-1) = \frac{3n-3}{2}$.

For even $n = 2k$, a possible model is made by a $(k-1)$-windmill plus one triangle "glued" by one side to the free side of a wing; the number of its edges is $e = 3(k-1) + 2 = 3k - 1 = \frac{3}{2}n - 1 = \frac{3n-2}{2} = \frac{3n-3}{2} + \frac{1}{2}$.

Denote by e_n the minimum number of edges sought after; these models show that $e_n \leqslant \lceil \frac{3n-3}{2} \rceil$, so $2e_n < 3n$.

On the other hand, consider such an extremal graph $G = (V, E)$, with $|V| = n > 3$ and $|E| = e_n$, and so $2|E| < 3|V|$. For any vertex v the number of incident edges is at least 2 (i.e. $\deg(v) \geqslant 2$), since if 0 – the vertex would be disconnected, while if 1 – that edge could not belong to any triangle. But then $\sum_{v \in V} \deg(v) = 2e_n < 3n$, hence there exists (at least) one vertex x with $\deg(x) = 2$. Denote by y, z the neighbours of x; then xy, xz and yz are edges.

If $[x, y, z]$ is the only triangle containing the edge yz, then remove the vertex x (together with edges xy and xz) and collapse y and z (thus removing the edge yz), to obtain a graph $G' = (V', E')$ with the stated properties, and $|V'| = |V| - 2$, $|E'| = |E| - 3$, while otherwise just remove the vertex x (together with edges xy and xz) to obtain a graph $G' = (V', E')$ with the stated properties, and $|V'| = |V| - 1$, $|E'| = |E| - 2$.

In both cases $|E| - |E'| \geqslant \frac{3}{2}(|V| - |V'|)$, and $2|E'| < 3|V'|$. Continue this procedure until left with one triangle, therefore $e_n \geqslant 3 + \frac{3}{2}(n-3) = \frac{3n-3}{2}$. Putting together the bounds on e_n we conclude that $e_n = \lceil \frac{3n-3}{2} \rceil$.

Problem 10. Let triangle ABC have $BC < AB$, and let points D on (AC), E on (AB) be such that $\angle DEB = \angle DCB$. It is given that point F lies in the interior of the quadrilateral $BCDE$, and the pairs of circumcircles of triangles BEF, CDF, respectively BCF, DEF, are tangent. Prove that points A, C, E, F, are concyclic.

[6] These are not the only ones having the stated properties and that number of edges – any "cactus", or tree of triangles, articulated on vertices, will do for odd n, while for even n we cannot avoid having one pair of triangles adjacent on one side.

Solution. We will show that point F lies on the segment BD, such that $\angle FEB = \angle DCF$ and $\angle DEF = \angle FCB$. Suppose F lies in the interior of triangle BDE and consider the common tangent HI for circles BEF, CDF; then $\angle BEF = \angle BFI = \angle HFG = \angle FCG < \angle FCD$, since point G (where BF meets again the circle CDF) lies in the interior of the small arc DF. Similarly, $\angle DEF < \angle BCF$; by summing inequalities obtained so far we get the contradiction $\angle BED < \angle BCD$. Moreover, from the above reasoning, $\angle BEF = \angle DCF$ and $\angle DEF = \angle BCF$, whence $\angle EFC + \angle EAC = 180°$, so the quadrilateral $ACFE$ is cyclic.

Problem 11. Let ABC be an acute-angled triangle, H its orthocenter and X any point in the plane. The circle of diameter HX meets the second time the line AH at point A_1, and the line AX at point A_2. Points B_1, B_2 and C_1, C_2 are defined in a similar way. Prove that the lines A_1A_2, B_1B_2, C_1C_2 are concurrent.

Solution. Denote by A', B', C' the feet of the altitudes from A, B, C.

We have $\angle B_1A_1C_1 = \angle B'HC = \angle C'HB = \angle A$, and analogues for B and C, hence triangles ABC and $A_1B_1C_1$ are similar.

Now, $\angle B_1A_1A_2 = \angle B_1HA_2 = \angle XAC$ (since $HA_2 \perp AX$ and $HB_1 \perp AC$), and analogues for B and C. Therefore, in the similarity of the two triangles, A_1A_2 corresponds to the isogonal of AX, and analogues for B and C.

Since AX, BX, CX are concurrent (at X!), it follows that their isogonals are also concurrent, hence A_1A_2, B_1B_2, C_1C_2 are concurrent.

Problem 12. For m and n odd integers larger than 1, prove that $2^m - 1$ does not divide $3^n - 1$.

Solution. Denote $M = 2^m - 1$. Assume $M | 3^n - 1$, then $(3^{(n+1)/2})^2 \equiv 3 \pmod M$, i.e. 3 is quadratic residue modulo M.

But $M \equiv 1 \pmod 3$ (since m odd), hence $(\frac{M}{3}) = 1$. Then, since M odd and $(M, 3) = 1$, we can use the quadratic reciprocity law for Jacobi symbols

$$\left(\frac{3}{M}\right) = \left(\frac{3}{M}\right)\left(\frac{M}{3}\right) = (-1)^{\frac{M-1}{2}\cdot\frac{3-1}{2}} = (-1)^{2^{m-1}-1} = -1,$$

i.e. 3 is not quadratic residue modulo M, contradiction.

Problem 13. A group of people is said to be *n-balanced* if in any subgroup of 3 people there exists (at least) a pair acquainted with each other, **and** if in any subgroup of n people there exists (at least) a pair not acquainted with each other.

i) Prove that the number of people in a n-balanced group has an upper bound. We may then denote by p_n the maximal possible number of people in a n-balanced group.

ii) Prove that $p_n \leqslant \frac{(n-1)(n+2)}{2}$.

iii) Compute, with proof, p_3, p_4 and p_5./

iv) Prove that $p_6 \leqslant 18$.

Solution. (D. Schwarz)[7] We will formulate the solution using graph-theoretical terminology, where G is a graph with the people as vertices, and edges between people acquainted with each other. Denote by δ_n, respectively Δ_n, the minimum, respectively maximum degree of the vertices of a n-balanced graph G (thus containing no complete K_n subgraph).

i) Then $|G| - \delta_n - 1 \leqslant n - 1$, since all non-neighbours of a vertex must be connected with edges. Therefore, $|G| \leqslant n + \delta_n$. We will use induction on n. Clearly, $p_2 = 2$. On the other hand, one must have $\Delta_n \leqslant p_{n-1}$, since the subgraph made by the neighbours of a vertex will need be $(n-1)$-balanced. Together, the two inequalities yield $|G| \leqslant n + \delta_n \leqslant n + \Delta_n \leqslant n + p_{n-1}$, so $|G| \leqslant n + p_{n-1}$, and therefore $p_n \leqslant n + p_{n-1}$ with equality needing, among others, $\delta_n = \Delta_n$, i.e. G to be (p_{n-1})-regular.

ii) Induction yields $p_n \leqslant n + \frac{(n-2)(n+1)}{2} = \frac{(n-1)(n+2)}{2}$.

iii) Now, $p_3 \leqslant 3 + p_2 = 5$, and a model for it is given by a graph on $\mathbb{Z}_5$ as vertices, with edges $\{\{i,j\}\,;\, i,j \in \mathbb{Z}_5, i - j \equiv \pm 1 \pmod 5\}$. This is seen as $G \equiv C_5$, a cycle of length 5.

The inequality above also yields $p_4 \leqslant 4 + p_3 = 9$, only that then G would be 5-regular, impossible, since in a graph the number of vertices of odd degree must

[7] This is actually related to the Ramsey numbers $R(p, q)$, as clearly $R(n, 3) = p_n + 1$. The upper bound $R(n, 3) = p_n + 1 \leqslant \frac{(n-1)(n+2)}{2} + 1 = \frac{n(n+1)}{2} = \binom{n+1}{2}$ has been established by Erdös, while the exact values $R(3, 3) = 6$, $R(4, 3) = 9$, $R(5, 3) = 14$ are known for some while. Even $R(6, 3) = 18$ is known, leading to $p_6 = 17$, and last known is $R(7, 3) = 23$.

be even. Therefore $p_4 \leqslant 8$, and a model for it is given by a graph on $\mathbb{Z}_8$ as vertices, with edges $\{\{i,j\}\,;\, i,j \in \mathbb{Z}_8, i-j \equiv \pm 1 \text{ or } \pm 2 \pmod 8\}$.

Finally, $p_5 \leqslant 5 + p_4 = 13$, and a model for it is given by a graph on $\mathbb{Z}_{13}$ as vertices, with edges $\{\{i,j\}\,;\, i,j \in \mathbb{Z}_{13}, i-j \equiv \pm 1, \pm 2, \pm 3 \text{ or } \pm 5 \pmod{13}\}$.

Checking that these model graphs are balanced is straightforward, but the model for $p_5 = 13$ will have to be argued comprehensibly in order to be accepted as solution.

iv) The inequality above yields now $p_6 \leqslant 6 + p_5 = 19$, only that then G would be 13-regular, impossible, since in a graph the number of vertices of odd degree must be even. Therefore, $p_6 \leqslant 18$.

A model to confirm this value however does not exist, as in fact it can be proven that $p_6 = 17$.

Alternative solution. i) and ii) We will use edges between people not acquainted with each other. The conditions translate into G not containing a K_3, **and** any n vertices having at least an edge between them.

Take any vertex v_1, with its n_1 neighbours; then any other vertex v_2, with its n_2 neighbours (among the remaining vertices after removing the first $1+n_1$), and so on, until vertex v_s, exhausting all vertices of G. Notice that neighbours of a common vertex v_k must be unconnected, otherwise a K_3 will be created.

Clearly, $s \leqslant n-1$, otherwise vertices $v_1, v_2, \ldots, v_n$, being unconnected by edges, will yield a contradiction. Also, we must have $n_k \leqslant n-k$, otherwise $n-k+1$ neighbours of v_k, together with $v_1, v_2, \ldots, v_{k-1}$, being unconnected by edges, will yield a contradiction. Therefore, the total number of vertices of G is at most

$$(1+(n-1)) + (1+(n-2)) + \cdots + (1+1) = \frac{(n-1)(n+2)}{2},$$

thus proving the assertions.

Problem 14. Consider the convex quadrilateral $ABCD$ with non-parallel opposite sides. Let O be the meeting point of lines AC and BD, P be the meeting point of lines AB and CD, and Q be the meeting point of lines AD and BC. Let R be the foot of the perpendicular from O onto PQ, and M, N, S, respectively T, the feet of the perpendiculars from R onto CD, BC, DA, respectively AB.

Prove that the points M, N, S and T are concyclic.

Solution. We have $\angle TSN = \angle TSA + \angle DSN$ and $\angle TMN = \angle RMN - \angle RMT$. Since quadrilaterals $RTAS$ and $RSNQ$ are cyclic, and also quadrilaterals $RPMT$ and $RNCM$ are cyclic, it follows that $\angle TSN + \angle TMN = 180^\circ + \angle PRA - \angle QRC$.

Let L be the meeting point of AC and PQ. Since (L, O, A, C) is a harmonic division, and $\angle LRO$ is a right angle, it is known that RO is the angle bisector of $\angle ARC$. Thus $\angle ARO = \angle ORC$, their complements are equal, and so $\angle PRA = \angle QRC$. It follows that $\angle TSN + \angle TMN = 180^\circ$, hence the quadrilateral $MNST$ is cyclic.

Problem 15. Given co-prime positive integers m, n, and integer s, compute the number of subsets $\{x_1, x_2, \ldots, x_m\} \subseteq \{1, 2, \ldots, m+n-1\}$ having

$$x_1 + x_2 + \cdots + x_m \equiv s \pmod n.$$

Solution. We claim the value of s is irrelevant, thus for any s the number of such subsets is $\frac{1}{n}\binom{m+n-1}{m}$.

We may and will assume $1 \leqslant x_1 < x_2 < \cdots < x_m \leqslant m+n-1$, whence $0 \leqslant y_k = x_k - k \leqslant n-1$ for all $1 \leqslant k \leqslant m$. The values $0 \leqslant y_1 \leqslant y_2 \leqslant \cdots \leqslant y_m \leqslant n-1$ constitute therefore a non-decreasing multiset of m elements between 0 and $n-1$, with $\sum_{k=1}^m y_k = \sum_{k=1}^m x_k - \frac{m(m+1)}{2}$.

Modulo n, the *shift* function $y_k \to y_k + 1$ adds m to $\sum_{k=1}^m y_k$ (with $n \equiv 0$), and since $(m,n) = 1$, the n^{th} iterate of the shift is the identity (and no lesser iterate is so). This is enough to establish a one-to-one correspondence between multisets with different sums of elements modulo n, which is what we claimed will be.

Alternative solution. We will prove through induction by k a more general statement:

Given positive integers k, n, and integer s, the number of subsets

$$\{x_1, x_2, \ldots, x_k\} \subseteq \{1, 2, \ldots, k+n-1\}$$

having

$$x_1 + x_2 + \cdots + x_k \equiv t \pmod n$$

is the same for all $t \equiv s \pmod{k}$.

The statement is trivial for $k = 1$. For $k > 1$ there is a one-to-one correspondence between sequences $1 \leqslant x_1 < x_2 < \cdots < x_k \leqslant k + n - 2$ with $\sum_{i=1}^{k} x_i \equiv s \pmod{n}$, and $2 \leqslant y_1 < y_2 < \cdots < y_k \leqslant k + n - 1$ with $\sum_{i=1}^{k} y_i \equiv s + k \pmod{n}$, realized by the *shift* function $x_i \to x_i + 1 = y_i$.

For $x_k = k + n - 1$ we have $x_{k-1} \leqslant (k-1) + n - 1$ and $\sum_{i=1}^{k-1} x_i \equiv s - (k + n - 1) \equiv s - (k-1) \equiv s' \pmod{n}$, while for $y_1 = 1$ we have $y_2 \geqslant 2$ and $\sum_{i=2}^{k} y_i \equiv (s + k) - 1 \equiv s + (k-1) \pmod{n}$. Use the *anti-shift* function $y_i \to y_i - 1 = z_{i-1}$ to get a sequence $1 \leqslant z_1 < z_2 < \cdots < z_{k-1} \leqslant (k-1) + n - 1$ with $\sum_{i=1}^{k-1} z_i \equiv \sum_{i=2}^{k}(y_i - 1) \equiv (s + (k-1)) - (k-1) \equiv s \equiv (s - (k-1)) + (k-1) \equiv s' + (k-1) \pmod{n}$.

We are now within the induction step for $k-1$, since $s' + (k-1) \equiv s' \pmod{(k-1)}$, hence sequences $1 \leqslant x_1 < x_2 < \cdots < x_k \leqslant k + n - 1$ with $\sum_{i=1}^{k} x_i \equiv s \pmod{n}$, and $1 \leqslant y_1 < y_2 < \cdots < y_k \leqslant k + n - 1$ with $\sum_{i=1}^{k} y_i \equiv s + k \pmod{n}$, have same cardinality. Iterative application of this result yields same cardinality for all sums $s + \mathcal{M}k$, i.e. for all $t \equiv s \pmod{k}$.

Now, for $k = m$, due to the fact that $(m, n) = 1$, the values $s + \mathcal{M}m$ range through all residues modulo n, thus establishing the claim.

REMARKS. This problem is reminiscent of a recent IMO Shortlist one:

Given prime $p > 2$ *and integer* s, *compute the number of subsets*

$$A = \{x_1, x_2, \ldots, x_p\} \subset \{1, 2, \ldots, 2p\}$$

having

$$x_1 + x_2 + \cdots + x_p \equiv s \pmod{p}.$$

A different approach is available here, based on generating functions. Let ω be a primitive p-root of unity, and consider the polynomial

$$P(x) = \prod_{i=1}^{2p}(x - \omega^i) = (x^p - 1)^2 = x^{2p} - 2x^p + 1.$$

On the other hand, the coefficient of x^p in $P(x)$ is

$$(-1)^p \sum_{A, |A| = p} \omega^{\sum_{a \in A} a} = -\sum_{t=0}^{p-1} \alpha_t \omega^t = -2, \quad \text{with} \sum_{t=0}^{p-1} \alpha_t = \binom{2p}{p},$$

where α_t is the number of subsets $A = \{x_1, x_2, \ldots, x_p\} \subset \{1, 2, \ldots, 2p\}$ having

$$x_1 + x_2 + \cdots + x_p \equiv t \pmod{p}.$$

Due to the fact that $1 + x + \cdots + x^{p-1}$ is irreducible (thus the minimal polynomial in $\mathbb{Z}[x]$ having ω as a root), it follows that $\alpha_t = \alpha_0 - 2$, for all $1 \leqslant t \leqslant p - 1$, hence

$$\alpha_0 = \frac{\binom{2p}{p} + 2(p-1)}{p}, \text{ while } \alpha_t = \frac{\binom{2p}{p} - 2}{p}, \text{ for all } 1 \leqslant t \leqslant p - 1.$$

Problem 16. For positive integer $n \geqslant 2$, prove that in any selection of at least $2^{n-1} + 1$ non-empty distinct subsets of $\{1, 2, \ldots, n\}$ there are three such that one of them is the union of the two other.

Solution. The proof is by induction. For $n = 2$, there are only $3 = 2^{2-1} + 1$ non-empty distinct subsets, and $\{1\} \cup \{2\} = \{1, 2\}$.

For $n \geqslant 2$ assume we have selected $2^n + 1$ non-empty distinct subsets of $\{1, 2, \ldots, n, n+1\}$. If at least $2^{n-1} + 1$ of them do not contain $n + 1$, then apply the induction hypothesis. If at least $2^{n-1} + 2$ of them do contain $n + 1$, then, by removing $n + 1$, there will remain at least $2^{n-1} + 1$ non-empty distinct subsets (since at most one subset is $\{n + 1\}$), then apply the induction hypothesis.

Therefore, the only case left is with exactly 2^{n-1} not containing $n + 1$, exactly 2^{n-1} containing $n + 1$ (but not as only element), and $\{n + 1\}$. By removing $n + 1$ we are left with 2^n non-empty subsets of $\{1, 2, \ldots, n\}$, and so two must be equal to some non-empty subset A. But then $B = A \cup \{n + 1\}$ is among our selection, and we are finished.[8]

Problem 17. For what positive integers n does there exist a permutation σ of $\{1, 2, \ldots, n\}$ such that the differences $|\sigma(k) - k|$, $1 \leqslant k \leqslant n$, are all distinct?

[8] The issue of the exact value of the minimum cardinality of such selections is moot. For $n = 2, 3, 4$, the stated bound can actually be achieved, but for larger n we know of no exact formula.

Solution. We claim such a permutation exists if and only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

To see the condition is necessary, notice that those n distinct differences must be, in some order, the numbers $0, 1, \ldots, n-1$, as $0 \leqslant |\sigma(k) - k| \leqslant n-1$, $1 \leqslant k \leqslant n$. Since $\sum_{k=0}^{n-1} k = (n-1)n/2$ we must have

$$\frac{(n-1)n}{2} = \sum_{k=1}^{n} |\sigma(k) - k| \equiv \sum_{k=1}^{n} (\sigma(k) - k) \equiv 0 \pmod 2.$$

Conversely, the permutation σ, having $\lceil \frac{3n}{4} \rceil$ as unique fixed point, and otherwise given by the $(n-1)$-cycle, for $n \equiv 0 \pmod 4$

$$\left(1, n, 2, n-1, \ldots, \frac{n}{4}, \frac{3n}{4}+1, \frac{n}{4}+1, \frac{3n}{4}-1, \ldots, \frac{n}{2}-1, \frac{n}{2}+1, \frac{n}{2}\right),$$

while for $n \equiv 1 \pmod 4$, denoting $n' = n - 1$

$$\left(1, n'+1, 2, n', \ldots, \frac{n'}{4}, \frac{3n'}{4}+2, \frac{n'}{4}+1, \frac{3n'}{4}, \ldots, \frac{n'}{2}, \frac{n'}{2}+1\right),$$

clearly has the desired property. [9]

Problem 18. Let ABC be a triangle, and $\mathcal{K}_a, \mathcal{K}_b, \mathcal{K}_c$ be the circles having its medians as diameters. Show that if two of these circles are tangent to the incircle of the triangle, then the third one is also tangent to the incircle.

Solution. Let $\mathcal{E}$ be the nine-point circle of the triangle, A_1 be the midpoint of the side BC, and A_2 be the foot of the altitude from A. By Feuerbach's Theorem, circle $\mathcal{E}$ is internal tangent to the incircle. On the other hand, circle $\mathcal{K}_a$ shares the chord A_1A_2 with circle $\mathcal{E}$ (except for the degenerate case when $A_1 \equiv A_2$).

Notice that $\mathcal{K}_a \equiv \mathcal{E}$ if and only if $\angle A = 90°$, since the antipodal of A_1 in $\mathcal{K}_a$ is the midpoint of the segment AH (where H is the orthocenter of the triangle). On the other hand, $A_1 \equiv A_2$ is equivalent to $AB = AC$.

We claim that in any situation other than the ones described above, the incircle is not tangent to $\mathcal{K}_a$. Recall that the segment A_1A_2 is a common chord for circles

[9] The differences between consecutive elements in the $(n-1)$-cycle decrement by 1 from $n-1$ to 1, with a gap at $\lfloor \frac{n-1}{2} \rfloor$, recuperated between the last and the first elements in the cycle.

$\mathcal{K}_a$ and $\mathcal{E}$; but the incircle lies entirely on one side of the line A_1A_2. On the same side, one of the circle segments from the circles $\mathcal{K}_a$ and $\mathcal{E}$ includes the other, since the two circles do not coincide. Consequently, the incircle cannot touch the arc of the outer segment, except when the tangency occurs at A_1 or A_2. But any of these implies that the incircle touches the side AB at the midpoint or at the foot of the altitude, hence ABC is an isosceles triangle – only that we assumed the contrary.

Thus circle $\mathcal{K}_a$ is tangent to the incircle if and only if $AB = AC$ or $\angle A = 90°$. Likewise, circle $\mathcal{K}_b$ is tangent to the incircle if and only if $BA = BC$ or $\angle B = 90°$. These two situations can simultaneously occur if and only if triangle ABC is equilateral, and then the third circle, $\mathcal{K}_a$ is tangent to the incircle.

Problem 19. Let $f(n)$ denote the maximum number of disjoint rectangles that the unit square I^2 in $\mathbb{R}^2$ can be partitioned into, such that any horizontal or vertical line intersects the interior of at most n rectangles. Show that

$$3 \cdot 2^{n-1} - 2 \leqslant f(n) \leqslant 3^n - 2.$$

(It is assumed that all the rectangles have sides parallel to the sides of I^2.)

Solution. Call the rectangles in a partition *tiles*. To obtain the lower bound, we proceed by induction. Clearly, $f(1) = 1$. Partition I^2 into 4 subsquares. Take the pattern achieving $f(n-1)$ and replicate it once into the upper left subsquare, and once into the lower right one. We now have a pattern that satisfies the condition for n. The number of its tiles is clearly $2f(n-1)+2$, hence $f(n) \geqslant 2f(n-1)+2$, a recurrence that simply leads to $f(n) \geqslant 3 \cdot 2^{n-1} - 2$.

In order to obtain the upper bound, define $f(m,n)$ to be the maximum number of tiles in a partition of some rectangle D, such that any horizontal line meets the interior of at most m tiles, while any vertical line meets the interior of at most n – call this the "(m,n) *condition*" (so $f(n,n) = f(n)$). [10] Clearly, $f(k,1) = f(1,k) = k$, for any k. We will show that $f(m,n) \leqslant 3^{(m+n)/2} - 2$, with the above cases easily seen to verify. We will proceed by induction on $m+n$. The result follows (easy calculation) from the following

[10] This is a case of proving a more general statement, in order for induction to work. Obviously, it is irrelevant that we work with initial squares or rectangles, of any dimensions.

LEMMA. $f(m,n) \leqslant 2f(m-1,n-1)+2+\max\{f(m,n-2),f(m-2,n)\}$.

Proof. Suppose we have a partition of D realizing $f(m,n)$. Let L be the widest tile that meets the left edge of D, let R be the widest tile that meets the right edge of D, and let $w(A)$ denote the width of rectangle A. Clearly, $w(L) < w(D)$ and $w(R) < w(D)$, since otherwise we can partition L or R with $m-1$ vertical segments, thus increasing $f(m,n)$. We now distinguish two cases.

(a) $w(L) + w(R) \leqslant w(D)$. We first cut D along the vertical lines at the right edge of L and left edge of R to get decompositions of three smaller rectangles. Call these pieces the *left, middle* and *right* pieces; the middle piece may be empty. Note that in this process, in general, new tiles may be created.

The left piece satisfies the $(m-1,n)$ condition since no tile touching the right edge of D comes farther left than R. Now, if we collapse L in the vertical direction, we obtain a rectangle satisfying the $(m-1,n-1)$ condition; hence the left piece has at most $f(m-1,n-1)+1$ tiles. Proceed in a similar way for the right piece. The middle piece satisfies the $(m-2,n)$ condition since no tile touching the left or right edge of D extends into it. Hence it contains at most $f(m-2,n)$ tiles.

Thus we have $f(m,n) \leqslant 2f(m-1,n-1)+2+f(m-2,n)$ in this case.

(b) $w(L) + w(R) > w(D)$. We similarly define the *left, middle* and *right* pieces; this time L and R both extend across the middle piece. As in the previous case, the number of tiles in the left or right piece is bounded by $f(m-1,n-1)+1$. The middle piece satisfies the (m,n) condition; however, if we collapse both the remnants of L and R, it will satisfy the $(m,n-2)$ condition. But the 2 collapsed remnants were already counted as part of L and R, in the left and right pieces.

Hence we have $f(m,n) \leqslant 2f(m-1,n-1)+2+f(m,n-2)$ in this case. Thus the Lemma is proved.[11] □

[11] The lower bound construction also extends to the general (m,n) condition, yielding $f(m,n) \geqslant 2f(m-1,n-1)+2$, which has the solution $f(m,n) \geqslant (m-n+3)\cdot 2^{n-1}-2$, given $m \geqslant n$ and $f(m,1)=m$. Furthermore, it is easy to prove $f(m,2)=2m$.

PROBLEMS AND SOLUTIONS

JUNIOR BMO SELECTION TESTS

Problem 1. Let p be a prime number, $p \neq 3$, and let a,b be integer numbers so that $p \mid a+b$ and $p^2 \mid a^3+b^3$. Show that $p^2 \mid a+b$ or $p^3 \mid a^3+b^3$.

Solution. Suppose that $p^2 \nmid a+b$. It suffices to prove that $p^3 \mid a^3+b^3$. Notice that $p^2 \mid (a+b)^3 - 3ab(a+b)$, we infer that $p \mid 3ab$. Since $p \neq 3$ is prime, it follows that $p \mid a$ or $p \mid b$. Since $p \mid a+b$, we get $p \mid a$ and $p \mid b$. Consequently, $p^3 \mid a^3$ and $p^3 \mid b^3$, implying $p^3 \mid a^3+b^3$.

Problem 2. Prove that for any positive integer n there exists a multiple of n whose decimal digits add up to n.

Solution. Let $n \geqslant 1$ and let 10^k, $k \in \mathbb{N}$. Consider all remainders of the numbers 10^k leave upon division by n. Since there are only finitely many residues, there exists $a = 0,1,\ldots,n-1$ so that $10^m \equiv a \pmod n$ for infinitely many values of $m \in \mathbb{N}$. Let $m_1 > m_2 > \cdots > m_n$ be a string of such m's. The number $A = 10^{m_1} + 10^{m_2} + \cdots + 10^{m_n}$ has n digits equal to 1, the remaining ones are all 0, so they add up to n. Moreover, $A \equiv na \equiv 0 \pmod n$, so $n \mid A$, which completes the proof.

Problem 3. Let ABC be an acute-angled triangle. Consider the equilateral triangle $A'UV$, with $A' \in (BC)$, $U \in (AC)$, $V \in (AB)$ such that $UV \parallel BC$. The points $B' \in (AC)$ and $C' \in (AB)$ are defined similarly. Show that the lines AA', BB' and CC' are concurrent.

Solution. Consider the equilateral triangle BCA_1, erected outwardly on the side BC. The points A, A', A_1 are collinear, through the homothety of center A which maps points U, V to B, C, respectively.

Since the lines AA_1, BB_1, CC_1 are concurrent at Fermat-Torricelli point of the triangle ABC, the conclusion follows.

Problem 4. Let ABC be a triangle and let D be the midpoint of the side BC. On the sides AB and AC there are points M, N respectively, other than the midpoints of these segments, so that $AM^2 + AN^2 = BM^2 + CN^2$ and $\angle MDN = \angle BAC$. Prove that $A = 90°$.

Solution. Let E and F be the midpoint of the sides AC and AB and let P be the reflected image of D across E.

Rewrite $AM^2 + AN^2 = BM^2 + CN^2$ as $(\frac{c}{2} + FM)^2 + (\frac{b}{2} - NE)^2 = (\frac{c}{2} - FM)^2 + (\frac{b}{2} + NE)^2$, to get $c \cdot FM = b \cdot NE$. Then $\frac{FM}{AC} = \frac{NE}{AB}$, so $\frac{FM}{FD} = \frac{NE}{EP}$. Since $\angle MFD = \angle NEP$, we get $\Delta MFD \sim \Delta NEP$, which implies $\angle MDF = \angle NPE$. On the other hand, $\angle MDN = \angle BAC = \angle FDE$, so $\angle MDF = \angle NDE$.

Now, the triangle NPD is isosceles and NE is a median in this triangle, so $NE \perp DP$, that is $A = 90°$.

Problem 5. Let $n \in \mathbb{N}$, $n \geqslant 2$ and let $a_1, a_2, \dots a_n$ be integer numbers such that $0 < a_k \leqslant k$, for all $k = 1, 2, \dots, n$. If $a_1 + a_2 + \dots + a_n$ is even, prove that that

$$a_1 \pm a_2 \pm \dots \pm a_n = 0,$$

for some choice of the signs "+" and "-".

Solution. Consider $A_{n-1} = a_n - a_{n-1}$. Since $a_n \leqslant n$ and $a_{n-1} \geqslant 1$, we have $A_{n-1} \leqslant n - 1$.

If $A_{n-1} = 0$, that is $a_{n-1} = a_n$, then $a_1 + a_2 + \dots + a_{n-2}$ is even and the claim reduces to the case of $n - 2$ numbers.

If $A_{n-1} > 0$, then $a_1 + a_2 + \dots + a_{n-2} + A_{n-1}$ is even and the claim reduces to the case of $n - 1$ numbers.

Problem 6. Consider an acute-angled triangle ABC, the height AD and the point E where the diameter through A of the circumcircle meets the line BC. Let M, N be the reflected images of D across the lines AC and AB. Show that $\angle EMC = \angle BNE$.

Solution. Notice that $AD = AN = AM$ and $\angle ANB = \angle AMC = 90°$, due to the reflections across AB and AC. The lines AD and AE are isogonal cevians, that is $\angle BAD = \angle EAC$. Then $\angle NAE = \angle NAB + \angle BAE = \angle BAD + \angle BAE = \angle EAC + \angle DAC = \angle EAC + \angle CAM = \angle EAM$ and consequently $\Delta NAE \equiv \Delta EAM$. It follows that $\angle ENA = \angle EMA$, so $\angle BNE = 90° - \angle ENA = 90° - \angle EMA = \angle EMC$, as desired.

Problem 7. Let $a_1, a_2, \dots, a_n$ be a sequence of integers such that a_k is the number of multiples of k in the sequence, for all $k = 1, 2, \dots, n$. Find all possible values of n.

Solution. Notice that $a_1 = n$, for 1 divides all a_j, and that all $a_j \leqslant n$.

Consider an n-by-n array whose (i, j)-entry is 1 if i divides a_j, and 0 otherwise. We use double counting. For each i, the sum of the entries of the i-th row is a_i, so the sum of all the array is $a_1 + \dots + a_n$.

On the other hand, for each j, the sum of the entries of the j-th column is the number of divisors of a_j, which is smaller than a_j, unless the latter is 1 or 2. So, the sum of all entries of the array is smaller than $a_1 + \dots + a_n$, unless $n = 1$ or $n = 2$.

Alternative solution. Recall that $a_1 = n$ and $a_i \leqslant n$, for all $i = 1, 2, \dots, n$. Assume $n > 3$. Since $a_{n-1} \geqslant 1$, the exists a multiple of $n - 1$, where $n - 1 > 1$, in the given sequence; let a_k, $k > 1$ be such a multipie. The condition $a_i \leqslant n$ shows that $a_k = n - 1$; in other words there, are $n - 1$ multiples of k in the sequence. Since n and $n - 1$ are coprime, k does not divide $a_1 = n$, so k divides $a_2, \dots, a_n$. But $k \geqslant 2$ and $k \mid a_n$, therefore $a_n > 1$. Thus n must occur at least twice in the sequence, so, beside a_1 we have $a_j = n$, $j > 1$. Hence $k \mid n$, a contradiction. As before, $n = 1$ or $n = 2$ are the only possible values.

Problem 8. Let $n \in \mathbb{N}^*$ and let $a_1, a_2, \dots, a_n$ be positive real numbers so that

$$a_1 + a_2 + \dots + a_n = \frac{1}{a_1^2} + \frac{1}{a_2^2} + \dots + \frac{1}{a_n^2}.$$

Prove that for any $m = 1, 2 \dots, n$, there exist m numbers among a_i whose sum is at least m.

Solution. It is clear that we need to prove that $a_1 + a_2 + \cdots + a_n \geqslant n$. Let us notice that this is enough: let $m = 1, 2, \dots, n$ and assume that any choice of m numbers among the a_i yields a sum less than m. In particular,

$$a_1 + a_2 + \cdots + a_m < m,$$

$$a_2 + a_3 + \cdots + a_{m+1} < m,$$

$$\vdots$$

$$a_n + a_1 + \cdots + a_{m-1} < m,$$

so $m(a_1 + a_2 + \cdots + a_n) < nm$, which is a contradiction.

Back to top, let g be the geometric mean of the numbers $a_1, a_2, \dots, a_n$ and suppose that $a_1 + a_2 + \cdots + a_n < n$. By the AM-GM inequality,

$$g \leqslant \frac{a_1 + a_2 + \cdots + a_n}{n} < 1,$$

while

$$1 > \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}}{n} \geqslant \frac{1}{g^2},$$

which yields $g > 1$, a contradiction.

Problem 9. Let a, b be real numbers with the property that the integer part of $an + b$ is an even number, for all $n \in \mathbb{N}$. Show that a is an even integer.

Solution. Let $[an + b] = 2x_n$, for all integers $n > 0$. Then

$$2x_n \leqslant an + b < 2x_n + 1, \tag{1}$$

$$2x_{n+1} \leqslant a(n+1) + b < 2x_{n+1} + 1. \tag{2}$$

Subtracting (1) from (2) we get $2(x_{n+1} - x_n) - 1 < a < 2(x_{n+1} - x_n) + 1$, for all $n > 0$. Since $2(x_{n+1} - x_n) - 1$ is an odd integer, it follows that all numbers $2(x_{n+1} - x_n) - 1$ must be equal, otherwise a would lie in two open intervals of length 2 whose left endpoints are at least 2 distance apart, which is impossible.

Hence $2(x_{n+1} - x_n) - 1 = 2s - 1$, $s \in \mathbb{Z}$, so $x_{n+1} - x_n = s$ and then $x_n = ns + p$, $p \in \mathbb{Z}, \forall n > 0$. Plug into (1) to get $2p - b \leqslant (a - 2s)n < 2p - b + 1, \forall n > 0$, so $a = 2s$, for otherwise the set of positive integers would have an upper bound. Observe that s is an integer, so a is an even integer, as required.

Problem 10. Ten numbers are chosen at random from the set $1, 2, 3, \dots, 37$. Show that one can select four distinct numbers from the chosen ones so that the sum of two of them is equal to the sum of the other two.

Solution. Consider all positive differences $a - b$ among all 10 numbers. Since there are $C_{10}^2 = 45$ positive differences and all belong to the set $1, 2, 3, \dots, 36$, at least two of them are equal. Let them be $a - b$ and $c - d$, with $a > c$. If a, b, c, d are all distinct, we are done; if not, then $b = c$, so $b = c$ is one of the 8 numbers which are neither the lowest nor the greatest number from the initial ones.

Now, observe that we have 45 positive differences and 36 possible values for them, so either 3 positive differences are equal or there are 9 pairs of equal positive differences.

The first case gives $a - b = c - d = e - f$, with $a > c > e$. Since we cannot have $b = c$, $b = e$ and $d = e$, we are done.

The second case gives at least one pair of positive differences in which case $b = c$ is excluded, as only 8 candidates for $b = c$ exist, so we are done.

Problem 11. Let a, b, c be positive real numbers with $ab + bc + ca = 3$. Prove that

$$\frac{1}{1 + a^2(b+c)} + \frac{1}{1 + b^2(c+a)} + \frac{1}{1 + c^2(a+b)} \leqslant \frac{1}{abc}.$$

Solution. Using the AM-GM inequality we derive $\frac{ab+bc+ca}{3} \geqslant \sqrt[3]{(abc)^2}$. As $ab + bc + ca = 3$, then $abc \leqslant 1$. Now

$$\sum \frac{1}{1 + a^2(b+c)} = \sum \frac{1}{1 + a(ab + ac)} = \sum \frac{1}{1 + a(3 - bc)}$$
$$= \sum \frac{1}{3a + (1 - abc)} \leqslant \sum \frac{1}{3a} = \frac{ab + bc + ca}{3abc} = \frac{1}{abc},$$

as required.

Problem 12. Find all primes p, q satisfying the equation $2p^q - q^p = 7$.

Solution. It is easy to observe that p is odd and $p \neq q$, in other words $p \geqslant 3$ and $(p, q) = 1$.

If $q = 2$, then $2^{p+1} = 7 + p^2$. The only solution is $p = 3$, as $2^{n+1} > 7 + n^2$, for all $n \geqslant 4$. For $q \geqslant 3$, by Little Fermat's Theorem we get $p \mid 2q - 7$ and $q \mid p + 7$. Set $p + 7 = kq$, $k \in \mathbb{N}^*$.

If $2q - 7 \leqslant 0$, we have $q = 3$ and $p \mid -1$, false

If $2q - 7 > 0$, then $2q - 7 \geqslant p$, so $2q \geqslant p + 7 \geqslant kq$, therefore $k = 1$ or $k = 2$. For $k = 1$ we obtain $p + 7 = q$, so $p \mid 2p + 7$. This implies $p = 7$ and then $q = 14$, false. Hence $k = 2$ and $p + 7 = 2q$. Suppose $p > q$; as $p, q \geqslant 3$ we get $q^p \geqslant q^p$ and then $7 = 2q^p - p^q \geqslant q^p \geqslant 3^3 = 27$, a contradiction. Thus $q > p$ and then $p + 7 = 2q > 2p$, which yields $p = 3$ or $p = 5$. For $p = 3$ we have $q = 5$, while $p = 5$ gives $q \mid 12$, with no solution.

To conclude, the solutions are $(p, q) = (3, 2), (3, 5)$.

Problem 13. Let d be a line and let M, N be two points on d. Circles $\alpha, \beta, \gamma, \delta$ centered at A, B, C, D are tangent to d in such a manner that circles α, β are externally tangent at M, while circles γ, δ are externally tangent at N. Moreover, points A and C lies on the same side of line d. Prove that if there exists a circle tangent to all circles $\alpha, \beta, \gamma, \delta$, containing all of them in the interior, then lines AC, BD and d are concurrent or parallel.

Solution. Let a, b, c, d be the radii of the circles $\alpha, \beta, \gamma, \delta$. It suffices to prove that $\frac{a}{b} = \frac{c}{d}$; in other words the ratio $\frac{a}{b}$ is constant while point M varies on line d.

Let R and S be the midpoints of the arcs determined by d on the fifth circle K, the one tangent simultaneously to $\alpha, \beta, \gamma, \delta$, and let N be on the same side of d as A. Denote by A_1 and B_1 the tangency point of α and β to K, respectively. Observe that points A_1, M, R are collinear – via the homothety which maps circle α onto circle K – and similarly points B_1, M, S are collinear. Since RS is a diameter of K, angles $\angle RA_1S$ and $\angle SB_1R$ are right. If lines B_1R and A_1S meet as point V, then M is the orthocenter of the triangle VRS. Notice that $d \perp RS$, hence $V \in d$; denote by O the intersection point of d and RS.

Lines A_1S and B_1R intersect the circles α and β at points U and Z respectively. Since $\angle RA_1S = \angle SB_1R = 90°$, the segments UM and ZM are diameters in circles α, β, so $\frac{a}{b} = \frac{UM}{ZM} = \frac{RO}{SO}$. The latter ratio is constant, as claimed.

Problem 14. Let $ABCD$ be a quadrilateral with no two opposite sides parallel. The parallel from A to BD meets the line CD at point F and the parallel from D at AC meet the line AB at point E. Consider the midpoints M, N, P, Q of the segments AC, BD, AF, DE respectively. Show that lines MN, PQ and AD are concurrent.

Solution. Let O be the midpoint of AD, R be the intersection point of lines AC and BD and S be the intersection point of lines AF and DE. Since N and Q are the midpoints of the sides DB and DE of the triangle DBE, we have $O \in NQ$ and similarly $O \in MP$. Moreover, as $DRAS$ is a parallelogram, the diagonal RS passes through the midpoint O of the other diagonal, AD. Now, apply Desargues Theorem for triangles NRM and SPQ, given that O lies simultaneously on lines NQ, MP, RS and we are done.

Problem 15. Let $m, n \in \mathbb{N}^*$ and $A = \{1, 2, \ldots, n\}$, $B = \{1, 2, \ldots, m\}$. A subset S of the set product $A \times B$ has the property that for any pairs $(a, b), (x, y) \in S$, then $(a - x)(b - y) \leqslant 0$. Show that S has at most $m + n - 1$ elements.

Solution. Consider a set S which satisfies all requirements. For each $i \in A = \{1, 2, \ldots, n\}$, define $B_i \subset B$ the set of all elements $j \in B$ for which the pair (i, j) belongs to the set S – notice that some subsets B_i can be empty. Counting all pairs in S over all second element in each pair, we have $|S| = |B_1| + |B_2| + \cdots + |B_n|$.

The main idea is to observe the chain of 'inequalities'

$$B_1 \leqslant B_2 \leqslant \cdots \leqslant B_n,$$

where by $X \leqslant Y$ we mean that $x \leqslant y$, for any $x \in X$ and $y \in Y$, X, Y being sets of integers. (This definition allows the empty set to occupy any position in this chain).

Since $B_1 \cap B_2 \cap \ldots \cap B_n = B = \{1, 2, \ldots, m\}$ and any two consecutive subsets B_i share in common at most one element, we get – by sieve thorem – that $|S| \leqslant m + n - 1$, as claimed.

Problem 16. Find all pairs of integers $(m,n), n,m > 1$ so that $mn-1$ divides n^3-1.

Solution. The solutions are (k,k^2) and (k^2,k), with $k>1$.

We have $mn-1 \mid (n^3-1)m - n^2(mn-1) = n^2-m$. On the other hand, $mn-1 \mid m(n^2-m)-(mn-1)n = n-m^2$.

If $n > m^2$, then $mn-1 \leqslant n-m^2 \leqslant n-1$, so $mn \leqslant n$, false.

If $n=m^2$, then obviously $m^3-1 \mid m^6-1$, so all pairs $(m,m^2), m>1$ are solutions.

If $n<m^2$, from $mn-1 \leqslant n^3-1$ we derive that $\sqrt{n} < m \leqslant n^2$. Then $mn-1 \leqslant m^2-n < m^2-1$, so $n<m$. If $n^2-m>0$, we obtain $mn-1 \leqslant n^2-m < n^2-1$, so $m<n$, a contradiction. Hence $n=m^2$, which holds, since $m^3-1 \mid m^3-1$, so all pairs $(n^2,n), n>1$ are also solutions.

Problem 17. Determine the maximum value of the real number k such that

$$(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c}-k\right) \geqslant k,$$

for all real numbers $a,b,c \geqslant 0$ with $a+b+c=ab+bc+ca$.

Solution. Observe that the numbers $a=b=2$, $c=0$ fulfill the condition $ab+bc+ca=a+b+c$. Plugging into the given inequality, we derive that $4\left(\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-k\right) \geqslant k$, hence $k \leqslant 1$.

We claim that the inequality holds for $k=1$, proving that the maximum value of k is 1. To this end, rewrite the inequality as follows

$$(ab+bc+ca)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c}-1\right) \geqslant 1 \Leftrightarrow$$

$$\sum \frac{ab+bc+ca}{a+b} \geqslant ab+bc+ca+1 \Leftrightarrow$$

$$\sum\left(\frac{ab}{a+b}+c\right) \geqslant ab+bc+ca+1 \Leftrightarrow \sum \frac{ab}{a+b} \geqslant 1.$$

Notice that $\frac{ab}{a+b} \geqslant \frac{ab}{a+b+c}$, since $a,b,c \geqslant 0$. Summing over a cyclic permutation of a,b,c we get

$$\sum \frac{ab}{a+b} \geqslant \sum \frac{ab}{a+b+c} = \frac{ab+bc+ca}{a+b+c} = 1,$$

as nedeed.

Alternative solution. The inequality is equivalent to the following:

$$S = \frac{a+b+c}{a+b+c+1}\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c}\right) \geqslant k.$$

Using the given condition, we get

$$\begin{aligned}\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c} &= \frac{a^2+b^2+c^2+3(ab+bc+ca)}{(a+b)(b+c)(c+a)} \\ &= \frac{a^2+b^2+c^2+2(ab+bc+ca)+a+b+c}{(a+b+c)(ab+bc+ca)-abc} \\ &= \frac{(a+b+c)(a+b+c+1)}{(a+b+c)^2-abc},\end{aligned}$$

hence

$$S = \frac{(a+b+c)^2}{(a+b+c)^2-abc}.$$

It is now clear that $S \geqslant 1$, and equality holds iff $abc=0$. Consequently, $k=1$ is the maximum value.

PROBLEMS AND SOLUTIONS

BALKAN MATHEMATICAL OLYMPIAD

Problem 1. Let ABC be a scalene acute-angled triangle with $AC > BC$. Let O be its circumcenter, H its orthocenter and F the foot of the altitude from C. Let P be the point (other than A) on the line AB for which $AF = PF$, and M the midpoint of the side AC. PH and BC meet at X, OM and FX meet at Y, and OF and AC meet at Z. Prove that points F, M, Y and Z are concyclic.

Solution. Since $OM \perp AC$, the conclusion is equivalent to $OF \perp FX$. This can be proven in many ways. A first possibility is the analytical approach: take coordinates $F(0,0), A(a,0), B(0,-b), C(0,c), a,b,c > 0$ and so $P(-a,0)$. Then $H(0,h)$, with $\frac{h}{a} = \frac{b}{c}$, hence $h = \frac{ab}{c}$. For the point $O\left(\frac{a-b}{2}, o\right)$ we have

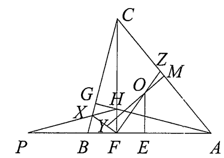
$$\left(\frac{a+b}{2}\right)^2 + o^2 = \left(\frac{a-b}{2}\right)^2 + (o-c)^2,$$

so $o = \frac{c^2-ab}{2c}$, while the point X is the intersection of the lines

$$HP{:}\ -\frac{x}{a} + \frac{y}{h} = 1, \qquad BC{:}\ -\frac{x}{b} + \frac{y}{c} = 1,$$

whence $X\left(\frac{b(ab-c^2)}{c^2-b^2}, \frac{bc(a-b)}{c^2-b^2}\right)$. Thus,

$$\overrightarrow{FO} \cdot \overrightarrow{FX} = \frac{b(ab-c^2)}{c^2-b^2}\,\frac{a-b}{2} + \frac{bc(a-b)}{c^2-b^2}\,\frac{c^2-ab}{2c} = 0.$$



Another possibility is to show that $\angle OFE = \angle CFX$ (E is the midpoint of the side AB), by means of trigonometry. We have

$$\operatorname{tg}\widehat{OFE} = \frac{OE}{EF} = \frac{OA\cos C}{\frac{c}{2} - a\cos B} = \frac{\cos C}{\sin C - 2\sin A\cos B} = \frac{\cos C}{\sin(B-A)},$$

since $\sin C - 2\sin A\cos B = \sin(A+B) - 2\sin A\cos B = \sin(B-A)$. Then, from Ceva's Theorem in trigonometric form applied in triangle PCF,

$$\frac{\sin\widehat{CFX}}{\sin\widehat{BFX}} \cdot \frac{\sin\widehat{BPX}}{\sin\widehat{CPX}} \cdot \frac{\sin\widehat{PCX}}{\sin\widehat{FCX}} = 1.$$

Now,

$$\angle BPX = \angle BAH = 90^\circ - B = \angle FCX,$$
$$\angle CPX = \angle CAH = 90^\circ - C,$$
$$\angle PCX = \angle PCF - \angle FCB = 90^\circ - A - (90^\circ - B) = B - A.$$

It follows

$$\operatorname{tg}\widehat{CFX} = \frac{\sin\widehat{CFX}}{\sin\widehat{BFX}} = \frac{\sin\widehat{CPX}}{\sin\widehat{PCX}} = \frac{\sin(90^\circ - C)}{\sin(B-A)} = \frac{\cos C}{\sin(B-A)},\ \text{q.e.d.}$$

The official solution replaced trigonometric calculations with computations based on triangle similitudes.

Problem 2. Does it exist a sequence $a_1, a_2, \ldots, a_n, \ldots$ of positive real numbers, which simultaneously satisfies

(i) $\sum\limits_{i=1}^{n} a_i \leqslant n^2$, for all positive integers n;

(ii) $\sum\limits_{i=1}^{n} \frac{1}{a_i} \leqslant 2008$, for all positive integers n?

Solution. The solution makes use of the AM-HM inequality and a computation of a lower bound

$$\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{a_i} \geqslant \frac{2^{2k}}{\sum_{i=2^k+1}^{2^{k+1}} a_i} > \frac{2^{2k}}{\sum_{i=1}^{2^{k+1}} a_i} \geqslant \frac{2^{2k}}{2^{2k+2}} = \frac{1}{4}.$$

It thus follows

$$\sum_{i=1}^{2^{2^n}} \frac{1}{a_i} = \frac{1}{a_1} + \sum_{k=0}^{n-1}\left(\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{a_i}\right) > \frac{1}{a_1} + \frac{n}{4},$$

whence the claim, since this lower bound may be made larger than 2008.

Alternative solution. Use the inequality

$$\sum_{i=1}^{n} \frac{1}{a_i} \geqslant \sum_{i=1}^{n} \frac{1}{2i-1}$$

which, in combination with the fact that the harmonic series is divergent, proves the claim. The previous inequality is obvious when $a_t \leqslant 2t-1$, for all $1 \leqslant t \leqslant n$, while otherwise, let us consider n fixed and replace step-by-step the terms of the sequence, such that at every replacement to obtain a sequence $(a'_t)_{1\leqslant t\leqslant n}$ which satisfies (i), while $\sum_{i=1}^{n} 1/a'_i \leqslant \sum_{i=1}^{n} 1/a_i$. The replacement is made using the algorithm:

If t exists such that $a_t > 2t-1$, then take the largest index, r, having this property, then the largest index, s, $s < r$, for which $a_s \leqslant 2s-1$ (such an index must exist, otherwise $\sum_{i=1}^{r} a_i > r^2$) and perform the replacement $a'_r = 2r-1$, $a'_s = a_s + a_r - (2r-1)$, and $a'_i = a_i$ for the rest.

Each such replacement decreases by at least 1 the number of indices t for which $a_t > 2t-1$, the sums $\sum_{i=1}^{t} a'_i$, $t < s$ or $t \geqslant r$, keep the same value, while

$$\sum_{i=1}^{t} a'_i = \sum_{i=1}^{r} a_i - (a_r + \cdots + a_{t+1}) \leqslant r^2 - ((2r-1) + \cdots + (2t+1)) = t^2,$$

for $s \leqslant t < r$. Moreover, the following inequality is immediate

$$\frac{1}{a'_s} + \frac{1}{a'_r} \leqslant \frac{1}{a_s} + \frac{1}{a_r},$$

therefore each replacement diminishes the sum $\sum_{i=1}^{n} \frac{1}{a'_i}$. Thus, by at most n loops in the algorithm, we obtain a sequence $b_1, b_2, \ldots, b_n$ with $b_i \leqslant 2i-1$, for all i and

$$\sum_{i=1}^{n} \frac{1}{a_i} \geqslant \sum_{i=1}^{n} \frac{1}{b_i} \geqslant \sum_{i=1}^{n} \frac{1}{2i-1}.$$

Alternative solution. Suppose the sequence $(x_n)_{n\geqslant 1} = (\sum_{i=1}^{n} 1/a_i)_{n\geqslant 1}$ is bounded; it follows due to monotony that it is convergent, hence a Cauchy sequence. This shows that taking, for example, $\varepsilon = \frac{1}{4}$, there exists $N \in \mathbb{N}$ such that $0 < x_n - x_m < \frac{1}{4}$, for all $n > m \geqslant N$. Using the Cauchy-Schwartz inequality we get

$$\sum_{i=N+1}^{2N} a_i \geqslant \frac{N^2}{\sum_{i=N+1}^{2N} \frac{1}{a_i}} = \frac{N^2}{x_{2N} - x_N} > 4N^2,$$

a contradiction with

$$\sum_{i=N+1}^{2N} a_i < \sum_{i=1}^{2N} a_i \leqslant (2N)^2 = 4N^2.$$

Problem 3. Let n be a positive integer. Rectangle $ABCD$, having the lengths of its sides $AB = 90n+1$ and $BC = 90n+5$, is partitioned in unit squares with sides parallel with the sides of the rectangle. Let S be the set of all points which are vertices of these unit squares. Prove that the number of distinct lines passing through at least two points of S is divisible by 4.

Solution. By doubling the dimensions, we may consider to have a system of coordinates with origin in the center of the rectangle and axes parallel to its sides. Therefore, $S = \{(2a+1, 2b+1) \mid a, b \in \mathbb{Z}, |a| \leqslant 45n, |b| \leqslant 45n+2\}$. Now, we partition the lines into four disjunct categories and show the number of lines within each category is divisible by 4.

The first category is made of lines parallel to one of the sides of the rectangle; their number is $90n+2+90n+6 = 4(45n+2)$.

The second category is made of lines not passing through the center of the rectangle (and not parallel to its sides) (Fig. 1); this class contains disjunct groups of four lines each d_1, d_2, d_3, d_4, symmetrical with respect to the center of the rectangle or to its symmetry axes.

Fig. 1

Fig. 2

The third category is made of lines passing through the center of the rectangle and some point of S situated within the square $\mathcal{P}$ of dimensions $(180n+2)\times(180n+2)$, with sides parallel to those of the rectangle, same center, and included in it (Fig. 2), except the diagonals of the square $\mathcal{P}$. This class contains disjunct groups of four lines each d_1', d_2', d_3', d_4', such that d_1' and d_2' are symmetrical with respect to the first diagonal, while d_3' and d_4' are the symmetrical of d_1' and d_2' with respect to the vertical axis.

Lastly, the fourth category is made of the diagonals of $\mathcal{P}$, as well as those lines passing through the center of the rectangle and some point of $S\setminus\mathcal{P}$; but no point of $S\cap\mathcal{P}$. These lines are characterized by having slopes of the form $\frac{p}{q}$, with $\frac{p}{q}$ irreducible fraction, $q\in\{90n+3, 90n+5\}$, p odd and $-(90n+1)\leqslant p\leqslant 90n+1$.

To count these lines, notice the set of odd integers co-prime with $90n+3$, situated in the interval $[-90n-1,-1]$ is in a one-to-one correspondence with the set of even integers co-prime with $90n+3$, situated in the interval $[2, 90n+2]$ through the law $x\leftrightarrow x+90n+3$. Thus, the set of irreducible fractions of the form $\frac{p}{90n+3}$, p odd, $|p|\leqslant 90n+1$ has the same cardinality with the positive integers co-prime with $90n+3$, situated in the interval $[1, 90n+2]$, i.e. $\varphi(90n+3)$. Similarly, the set of irreducible fractions of the form $\frac{p}{90n+5}$, p odd, $|p|\leqslant 90n+3$ has $\varphi(90n+5)-2$ elements (missing are $90n+3$ and $-(90n+3)$). Adding to the sum the two diagonals of $\mathcal{P}$ as well, it follows that the fourth category contains $\varphi(90n+3)+\varphi(90n+5)$ lines.

Let's show numbers $\varphi(90n+3)$ and $\varphi(90n+5)$ are divisible by 4. Since 3 and $30n+1$ are co-prime, $\varphi(90n+3)=\varphi(3)\varphi(30n+1)=2\varphi(30n+1)$ and $\varphi(30n+1)$ is even, while $\varphi(90n+5)$ is divisible by $\varphi(5)=4$, and the proof

is done.

Problem 4. Let c be a positive integer. The sequence $(a_n)_{n\geqslant 1}$ is defined by $a_1=c$, $a_{n+1}=a_n^2+a_n+c^3$, for all positive integers n. Determine all values of c for which there exist integers $k\geqslant 1$, $m\geqslant 2$ such that $a_k^2+c^3$ be the power m of some integer.

Solution. From the recurrence relation we get $a_n^2+c^3=a_{n+1}-a_n$, and then, for all $n\geqslant 2$

$$a_{n+1}-a_n=a_n^2-a_{n-1}^2+a_n-a_{n-1}=(a_n-a_{n-1})(a_n+a_{n-1}+1).$$

We will show the factors a_n-a_{n-1} and $a_n+a_{n-1}+1$ are co-prime. In effect, if they would share a common prime factor p, then $p\mid(a_n-a_{n-1}+a_n+a_{n-1}+1)=2a_n+1$ and $p\mid(a_n+a_{n-1}+1-a_n+a_{n-1})=2a_{n-1}+1$. From the relation

$$2(2a_n+1)=(2a_{n-1}+1)^2+(4c^3+1)$$

it now follows $p\mid(4c^3+1)$, while from the same relation for $n-1$ it follows $p\mid(2a_{n-2}+1)^2$, hence $p\mid(2a_{n-2}+1)$. We deduce that p divides all numbers of the form $2a_s+1$, $1\leqslant s\leqslant n$; in particular, $p\mid 2a_1+1=2c+1$. But a routine reasoning shows that $(2c+1, 4c^3+1)=1$, hence the assumption was false.

From the above it follows that, if $a_k^2+c^3=a_{k+1}-a_k$, for $k\geqslant 2$, is equal to the power $m, m\geqslant 2$ of some integer, then also $a_k-a_{k-1}=a_{k-1}^2+c^3$ is equal to the power $m, m\geqslant 2$ of some integer.

Therefore, to fulfill the requirement in the problem, it is necessary (and sufficient) that $a_1+c^3=c^2(c+1)$ be equal to the power $m, m\geqslant 2$ of some integer. Since c^2 and $c+1$ are co-prime, while c^2 is a perfect square, the requirement in the problem is fulfilled if and only if $c+1$ is a perfect square.[12]

[12]From the above it also follows that, if a term a_k of the sequence fulfills the requirement, then $a_k^2+c^3$ is a perfect square. But, for $k\geqslant 2$, $a_k^2<a_k^2+c^3<(a_k+1)^2$, hence the only term that may fulfill the requirement is a_1.

PROBLEMS AND SOLUTIONS

ROMANIAN MASTER IN MATHEMATICS COMPETITION

Problem 1. Let ABC be an equilateral triangle. P is a variable point internal to the triangle and its perpendicular distances onto the sides are denoted by a^2, b^2 and c^2 for positive real numbers a, b and c. Find the locus of points P so that a, b and c can be the sides of a non-degenerate triangle.

Solution. The locus is the interior of the incircle of triangle ABC.

To prove this, embed the equilateral triangle in the Cartesian space $Oxyz$, as the set in the plane $x + y + z = 1$ described by $x, y, z \geqslant 0$. Let the feet of the perpendiculars from P onto BC and CA be D and E respectively, and let the feet of the perpendiculars from P onto the planes OBC and OCA be Q and R respectively. Then triangles PQD and PRE are similar, so $PQ : PR = PD : PE$; i.e. $x : y = a^2 : b^2$, where (x, y, z) are coordinates of P. In the same way we get $y : z = b^2 : c^2$, so we have $(a^2 : b^2 : c^2) = (x : y : z)$.

Now if a, b and c are the sides of a triangle, the Heron's formula states that the square of the area of that triangle is

$$\frac{1}{16}(a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

So, this quantity is positive. The reverse is also true.

Multiplying the expression out, this means that a, b and c are the sides of a triangle if and only if

$$2\sum b^2c^2 - \sum a^4 > 0.$$

Since a^2, b^2, c^2 are proportional to x, y, z, it follows that a, b and c are the sides of a triangle if and only if

$$2(x^2 + y^2 + z^2) < (x + y + z)^2 = 1.$$

So, the required locus of points is the intersection of the solid sphere $x^2+y^2+z^2 < 1/2$ with the plane $x + y + z = 1$; that is the interior of the inscribed circle of the equilateral triangle.

Alternative solution. Using a^2, b^2, c^2 as barycentric coordinates for P, in an equilateral triangle of circumradius 1, one can calculate the distance from P to the incenter I, reducing thus the problem to an algebraic one. In fact, one can see the similarity to the above solution.

Problem 2. Given positive integer $a > 1$, prove that any positive integer N has a multiple in the sequence

$$(a_n)_{n\geqslant 1}, \quad a_n = \left\lfloor \frac{a^n}{n} \right\rfloor.$$

Solution. In what follows, all literals will represent non-negative integers. The solution makes use of specific values for n, carefully chosen to facilitate the computation of the *floor* function.

Clearly, there exist $e \geqslant 0$, $q \geqslant 1$ and

$$M = a^{a^e - e}q, \quad \gcd(q, a) = 1,$$

such that M is a multiple of N.

Let us consider values $n = a^e p$, with p prime, $p > M$. Then, by Fermat's Theorem ($p > M \geqslant a$, so $\gcd(a, p) = 1$)

$$a^{a^e(p-1)} - 1 = (a^{p-1})^{a^e} - 1 \equiv 0 \pmod p, \quad \text{so } a^n = a^{a^e}kp + a^{a^e},$$

therefore, as $n = a^e p > a^e M \geqslant a^{a^e}$

$$a_n = \left\lfloor \frac{a^n}{n} \right\rfloor = a^{a^e - e}k.$$

On the other hand, $kp = a^{a^e(p-1)} - 1$. Assuming $p - 1 = m\varphi(q)$ we have $a^{\varphi(q)} \equiv 1 \pmod q$[13], therefore $kp \equiv 0 \pmod q$, so q divides kp. But $p > M > q$, so $\gcd(q, p) = 1$, hence q divides k, so M (and *a fortiori* N) divides a_n.

We are left to prove that we can find such $p - 1 = m\varphi(q)$, that is, $p > M$ must belong to the arithmetic sequence of first-term 1 and ratio $\varphi(q)$.

[13] φ is the Euler *totient* function, and $\gcd(q, a) = 1$

The existence of such p is guaranteed by Dirichlet's Theorem[14] and that should suffice in an international math competition.

REMARKS. We will however, for self-containment, present a proof for this particular case of Dirichlet's Theorem. [15]

An arithmetical sequence of first-term 1 *and ratio* r *contains infinitely many primes (assume* $r > 2$*, as* $r = 1$ *or* $r = 2$ *makes it trivially true).*

We will denote by d, $1 \leqslant d < r$, any (proper) divisor of r. Let us consider the polynomial $X^r - 1 \in \mathbb{Z}(X)$, factored in irreducible polynomials. Its roots (the *r-roots of unity*) are

$$\cos\frac{2k\pi}{r} + \mathrm{i}\sin\frac{2k\pi}{r}, \qquad \text{with } 1 \leqslant k \leqslant r,$$

and, for $k = 1$, the main *primitive* r-root of unity ζ cannot be the root of any polynomial $X^d - 1$. Therefore, ζ must be root of an irreducible factor $f(X)$ for $X^r - 1$, which cannot be a factor for any $X^d - 1$.[16] Now

$$f(X) \text{ divides } \frac{X^r - 1}{X^d - 1} \text{ for all } d, \text{ and } f(X) = \prod_{i=1}^{\deg f} (X - z_i),$$

with z_i among the r-roots of unity, so $|z_i| = 1$. Therefore, for any $n > 2$

$$|f(n)| = \prod_{i=1}^{\deg f} |n - z_i| \geqslant \prod_{i=1}^{\deg f} |n - |z_i|| = (n-1)^{\deg f} > 1.$$

Assume now there are only finitely many such primes q, and take $n = r\prod q$.[17] As $|f(n)| > 1$, there exists p prime, dividing $f(n)$, and therefore dividing $\frac{n^r-1}{n^d-1}$ for all d. We then cannot have p dividing $n^d - 1$ for any d, because

$$X^{\frac{r}{d}} - 1 = (X-1)P(X), P(X) = (X-1)Q(X) + R, R = P(1) = \frac{r}{d},$$

[14] Dirichlet's Theorem asserts the existence of infinitely many primes in an arithmetic sequence of co-prime first-term and ratio.

[15] This effort is an improvement on a proof by A. Rotkiewicz.

[16] In fact (not needed here), all primitive roots, for $\gcd(k, r) = 1$, are the roots of a **same** irreducible factor $\Phi_r(X)$, of degree $\varphi(r)$, which is the *cyclotomic polynomial* of order r. Then $X^r - 1 = \prod_{d|r} \Phi_d(X)$, the product of the (irreducible) cyclotomic polynomials.

[17] By definition $\prod q := 1$ if no such primes were to be selected.

so $\frac{n^r-1}{n^d-1} = P(n^d) = (n^d - 1)Q(n^d) + \frac{r}{d}$, while clearly $n^d - 1$ and $\frac{r}{d}$ are co-prime (as r divides n), therefore p cannot divide $\frac{r}{d}$.

This shows that $n^r \equiv 1 \pmod p$ and $n^d \not\equiv 1 \pmod p$ for any d, so $r = \mathrm{ord}_p(n)$. But $n^{p-1} \equiv 1 \pmod p$ (by Fermat's Theorem), so we must have r dividing $p - 1$, that is, p belongs to the stated arithmetical sequence. However, $p \neq q$ for any q considered in the above, as $\gcd(p, n) = 1$, and thus we have found yet another such prime, contradiction. □

Problem 3. Prove that any one-to-one surjective function $f : \mathbb{Z} \to \mathbb{Z}$ can be written as $f = u + v$ where $u, v : \mathbb{Z} \to \mathbb{Z}$ are one-to-one surjective functions.

Solution. (D. Schwarz) To find u, v such that $f = u + v$ it is enough to consider the case $f =$ identity on $\mathbb{Z}$. For that it suffices to write the above relation as $\mathrm{id}_{\mathbb{Z}} = u \circ f^{-1} + v \circ f^{-1}$. Consider the following well-ordering of the nonzero integers: $\mathbb{Z}^* = \{1, -1, 2, -2, \ldots, n, -n, \ldots\}$.

Build the following table

Step	A	#	B
1	1	+1	2
2	−1	−2	−3
3	−2	−3	−5
4	3	+4	7
⋮	⋮	⋮	⋮
k	a_k	$\mathrm{sign}(a_k) \cdot k$	$b_k = a_k + \#(k)$
⋮	⋮	⋮	⋮

The inductive rule in completing the table is as follows: at step 1 write 1, the first in the ordering of $\mathbb{Z}^*$, in column A, in column # put the number of the step, that is 1, with the sign from A, and in column B the sum from A and #. Suppose now that row of step i has been completed. Write on row $i + 1$ in column A the first integer in the ordering of $\mathbb{Z}^*$ that has not yet been used in A nor B, in column # the number $i + 1$ with the sign given by that of the number just written in A, and in B the sum of A and #.

It is easy to see that in this manner we get an infinite array where $A \cup B = \mathbb{Z}^*$ and $A \cap B = \emptyset$, while elements in A and B do not repeat.

Define now $u(0) = v(0) = 0$ and for $x \in \mathbb{Z}$

- for $x = a_i \in A$ (meaning that x is in column A and row i), take $u(x) = -\#(i), v(x) = b_i$;
- for $x = b_j \in B$, take $u(x) = \#(j), v(x) = a_j$.

Obviously, u and v are both bijections from $\mathbb{Z}$ to $\mathbb{Z}$ and $\mathrm{id}_{\mathbb{Z}} = u + v$.[18]

Problem 4. Prove that from among any $(n+1)^2$ points inside a square of side length positive integer n, one can pick three, such that the triangle determined by them has area no more than $\frac{1}{2}$.

Solution. Although the topic of the problem may somehow appear familiar, the solution involves a novel and ingenious mix of ideas, centered around estimating areas of triangles using simple convexity inequalities.

Denote by $A = n^2$ the area of the square, by $P = 4n$ the perimeter of the square, and by $N = (n+1)^2$ the number of points. The convex hull of the set of N points will be a convex k-gon (contained in the given square), $3 \leqslant k \leqslant N$, with $N - k$ points in its interior (if any three points are collinear, they will determine a triangle of area 0, thus rendering the result trivially).

We will make use of the following folklore result:

Any triangulation of a (convex) k-gon, using m = N - k interior points, is made of t = (k - 2) + 2m = 2(N - 1) - k triangles.[19]

As the area of the convex hull k-gon is at most A, it follows, using an *averaging* argument, that there will exist a triangle Δ_f of area at most

$$\frac{A}{t} = \frac{A}{2(N-1)-k} = f(k).$$

On the other hand, as the perimeter of the convex hull k-gon is at most P, one can find a pair of consecutive sides, be them **a**, **b**, of lengths a, b, such that $\frac{a+b}{2} \leqslant \frac{P}{k}$

[18]The result follows immediately from Lindenbaum's Theorem, stating that for any countable infinite additive group G, every function $u : \mathbb{N} \to G$ is a sum of three one-to-one surjective functions, $u = u' + u'' + u'''$ (but two only are not always enough). Just take $u \equiv 0$, and then it's simple.

[19]The total sum of angles for the t triangles is $t\pi$; but the vertices contribute $(k-2)\pi$, while the interior points contribute $2m\pi$, therefore $t = (k-2) + 2m$.

(this also is an *averaging* argument). Now, the area of the triangle Δ_g determined by **a**, **b**, is

$$\frac{1}{2}\, ab \sin\angle(\mathbf{a},\mathbf{b}) \leqslant \frac{1}{2}\Big(\frac{a+b}{2}\Big)^2 \leqslant \frac{P^2}{2k^2} = g(k).$$

Clearly, the bounds for the areas of triangles Δ_f, Δ_g depend on k, but $f(k)$ is increasing, while $g(k)$ is decreasing, therefore the worst case occurs for the value calculated in k_0 where the graphs of f and g meet

$$\frac{A}{2(N-1)-k_0} = \frac{P^2}{2k_0^2},$$

so $k_0^2 = 16(n+1)^2 - 16 - 8k_0$, hence $k_0 = 4n$. Both formulae f and g, calculated in k_0, yield the value $\frac{1}{2}$, as required.

REMARKS. One can improve on the bound given by $g(k)$; in fact it may be proven that a triangle Δ_g of area at most $\frac{P^2}{2k^2}\sin\frac{2\pi}{k}$ can be found. However, the minimum value offered by $f(k)$ is greater than $\frac{1}{2}(\frac{n}{n+1})^2$, which converges to $\frac{1}{2}$ when n grows large, thus thwarting any attempt to improve on the $\frac{1}{2}$ bound. The issue is to improve on the bound given by $f(k)$, but it is difficult to find efficient ways to bound from above the size of a least-area triangle for small k.

The author is far from claiming the result is tight (for large n), although better estimates appear elusive; however the naive attempt to use the pigeonhole principle in its simplest form (partition the side-n square into n^2 unit squares; then for any $2n^2 + 1$ points inside the square there will exist three within a unit square, thus determining a triangle of area at most $\frac{1}{2}$), necessitates almost twice as many points as those afforded in the problem (except for $n = 2$, when $2 \cdot 2^2 + 1 = (2+1)^2$). On the other hand, for $n = 1$, the result is best possible!

Moreover, using the $\frac{P^2}{2k^2}\sin\frac{2\pi}{k}$ bound for Δ_g, one can prove for $n = 2$ that there exists a triangle of area at most $\frac{4}{9}$ (the critical point k_0 is moving from value 8 to 7, when the correct answer is given by $f(7) = \frac{4}{9}$), a better bound than anything found in the literature!

PROBLEMS AND SOLUTIONS

IMAR MATHEMATICAL COMPETITION

Problem 1. For real numbers $x_i > 1$, $1 \leqslant i \leqslant n$, $n \geqslant 2$, such that

$$\frac{x_i^2}{x_i - 1} \geqslant S = \sum_{j=1}^{n} x_j, \quad \text{for all } i = 1, 2, \ldots, n$$

find, with proof, $\sup S$.

Solution. (D. Schwarz) For $n = 2$, S is unbounded to the right, since for the pairs $(v, \frac{v}{v-1})$, where $v > 1$, we have $S = \frac{v^2}{v-1}$, and then $\lim\limits_{v\to 1} \frac{v^2}{v-1} = \infty$.

For $n > 2$, we will prove the tight upper bound $S \leqslant \frac{n^2}{n-1}$, with equality iff all $x_i = \frac{n}{n-1}$.

Consider the function $f : (1, +\infty) \to [4, +\infty)$, defined by $f(x) = \frac{x^2}{x-1}$. It is straightforward to prove that f is strictly decreasing on $(1, 2]$. Let

$$M = \max_{i=1}^{n} x_i;$$

then $\frac{M^2}{M-1} \geqslant S > M + (n-1)$, hence $M < \frac{n-1}{n-2} \leqslant 2$, so $x_i \in (1, 2)$ for all i.

Now, for $m > 1$, either $x_i \geqslant m$ for some i, and then $S \leqslant \frac{x_i^2}{x_i-1} \leqslant \frac{m^2}{m-1}$ (according to the monotonicity of f), or $x_i \leqslant m$ for all i, and then clearly $S \leqslant nm$. Solving the equation $\frac{m^2}{m-1} = nm$, obtained by equaling the two possible upper bounds for S, yields as unique solution $m_0 = \frac{n}{n-1}$, therefore, in all cases, $S \leqslant \frac{m_0^2}{m_0-1} = nm_0 = \frac{n^2}{n-1}$, with equality iff all $x_i = \frac{n}{n-1}$.[20]

[20]Since $f(x) \geqslant S$ leads to a degree 2 trinomial, the same result may be obtained through the study of the discriminant, and of the positions of x_i's relative to its roots. However, the problem could be generalized to some more general function f, with specific characteristics.

Problem 2. Denote by $\mathcal{C}$ the family of all configurations C of $N > 1$ distinct points on the sphere S^2, and by $\mathcal{H}$ the family of all closed hemispheres H of S^2. Compute

$$\max_{H\in\mathcal{H}} \min_{C\in\mathcal{C}} |H \cap C|, \qquad \min_{H\in\mathcal{H}} \max_{C\in\mathcal{C}} |H \cap C|,$$

$$\max_{C\in\mathcal{C}} \min_{H\in\mathcal{H}} |H \cap C| \quad \text{and} \quad \min_{C\in\mathcal{C}} \max_{H\in\mathcal{H}} |H \cap C|.$$

Solution. Denote by $\overline{H} := S^2 \setminus H$.

Clearly,

$$\max_{H\in\mathcal{H}} \min_{C\in\mathcal{C}} |H \cap C| = 0,$$

since for any H we may use C_- made of N points bunched together on $\overline{H}$.

Similarly, clear

$$\min_{H\in\mathcal{H}} \max_{C\in\mathcal{C}} |H \cap C| = N,$$

since for any H we may use C_+ made of N points bunched together on H.

It is rather more difficult to show that

$$\max_{C\in\mathcal{C}} \min_{H\in\mathcal{H}} |H \cap C| = \left\lfloor \frac{N}{2} \right\rfloor,$$

since for any C, considering a great circle Γ which passes through no point of C, by pigeonhole principle we have that for H, one of the two induced closed hemispheres, $|H \cap C| \leqslant \lfloor \frac{N}{2} \rfloor$. On the other hand, for C_ω defined as being made of pairs of antipodal points (plus one more point when N odd), any H contains at least $\lfloor \frac{N}{2} \rfloor$ points of C, as $|\overline{H}| + |H| = N$, and $|\overline{H}| \leqslant |H| + 1$,[21] yield $2|H| + 1 \geqslant N$, so $|H| = \left\lfloor \frac{2|H|+1}{2} \right\rfloor \geqslant \left\lfloor \frac{N}{2} \right\rfloor$.

Finally, it is the most difficult to show that

$$\min_{C\in\mathcal{C}} \max_{H\in\mathcal{H}} |H \cap C| = \left\lfloor \frac{N+3}{2} \right\rfloor.$$

For any C, considering a great circle Γ which passes through two points of C, by pigeonhole principle we have that for H, one of the two induced closed hemi-

[21]Points in $\overline{H}$ are matched by their antipodes in H, with (maybe) one unmatched when N odd.

spheres, $|H \cap C| \geqslant \left\lfloor \frac{(N-2)+1}{2} \right\rfloor + 2$. On the other hand, let us define configuration C_N for N even.

Consider the tropic-circles on S^2, and on each a regular $\frac{N}{2}$-gon. Now

• for $\frac{N}{2}$ odd, take the two polygons both with a vertex on the 0 meridian, making a regular $\frac{N}{2}$-prism;

• for $\frac{N}{2}$ even, take one polygon with a vertex on the 0 meridian, and the other with a vertex on the $\frac{\pi}{N}$ meridian, making a twisted regular $\frac{N}{2}$-prism.

One can check that, for this configuration, any hemisphere H contains at most $\frac{N}{2} + 1 = \left\lfloor \frac{N+3}{2} \right\rfloor$ points of C_N.[22] This is more easily seen if the points lying on one tropic-circle are projected onto the equatorial plane and colored red, while the antipodes of the points lying on the other tropic-circle are projected onto the equatorial plane and colored blue. We can now study the effect of planes through the center of the sphere, inducing all possible hemispheres.

For N odd, augment C_{N-1} with any other more point, to obtain C_N. Then, according to the results above, any hemisphere H contains at most $\left(\frac{N-1}{2} + 1\right) + 1 = \left\lfloor \frac{N+3}{2} \right\rfloor$ points of C_N.

Problem 3. Prove that among $N \geqslant 2n - 2$ integers, of absolute value not higher than $n > 2$, and of absolute value of their sum S less than $n - 1$, there exist some of sum zero.

Show that for $|S| = n - 1$ this is not anymore true, and neither for $N = 2n - 3$ (when even for $|S| = 1$ this is not anymore true).

Solution. (D. Schwarz) We may assume that the given integers are nonzero (since any equal to zero may be selected), and $S \geqslant 1$ (since if negative, we may multiply all numbers by -1, while if zero, itself may be selected). Let us denote

$$-n \leqslant -m = n_0 \leqslant n_1 \leqslant \cdots \leqslant n_\nu < 0 < p_\pi \leqslant \cdots \leqslant p_1 \leqslant p_0 = M \leqslant n$$

with $\nu \geqslant 0$, $\pi \geqslant 0$ (it is clear that from the conditions it follows that not all of the integers have the same sign). Denote $\nu^* = \nu + 1$, $\pi^* = \pi + 1$, so $\nu^* + \pi^* = N$.

[22] Notice that configuration C_ω is not acceptable here, since a great circle Γ passing through two not opposite points of C also contains their antipodes, so any of its two induced closed hemispheres contains at least $\frac{N-4}{2} + 4 = \frac{N}{2} + 2$ points of C.

Obviously, we should assume $M + m \leqslant 2n - 1$, since $M + m = 2n$ would lead to $M = m = n$, and then $n_0 + p_0 = 0$.

The method applied is to consider some interval I of integer numbers, start by choosing some given integer in I (as first sum σ_1), then build sums σ_s by adding the other given integers one by one (in some prescribed order), such that all these sums are contained in I. The idea is to pack these sums as tightly possible in a conveniently used interval. If the total number of these sums is greater than the number of integers contained in I, then, by pigeonhole principle, there will be two of them equal. Then their difference (in itself a sum of the initially given integers) will be zero. Call this reasoning principle *Lemma*.[23]

For once, in order to compress the solution, we will abdicate from our expositive, self-explanatory method, and present a compact proof; different parts of it might have been done in a different, simpler way.

Denote $\delta = (2n-1) - (M+m)$, so, according to the above, $\delta \geqslant 0$. Consider the interval $I = [-m+1, M-1+\delta]$, containing $2n-2$ integers, including 0. Assume there exists a given integer $x \in I$. Start with $\sigma_1 = x$, and add p_i's, in any order, as long as $\sigma_s \in I$ or we run out of positives, then add n_j's, in any order, as long as $\sigma_s \in I$ or we run out of negatives, then repeat, until this algorithm stops. Notice that $\sigma_s < 0$ allows adding a p_i ($\sigma_s + p_i \leqslant -1 + M$), while $\sigma_s > 0$ allows adding a n_j ($\sigma_s + n_j \geqslant 1 - m$). If ever we reach $\sigma_s = 0$, we found a zero sum.

The outcome is that we end up with all possible N sums in I (since if we run out of p_i's or n_j's, we continue with the others until we reach $\sigma_N = S \leqslant n - 2 \leqslant 2n - 2 - m = M - 1 + \delta$), therefore, either one of them is 0, or else they can take at most $2n - 3 < N$ values, and so Lemma applies.

Now, if $M + m \leqslant 2n - 2$, when $\delta \geqslant 1$, we can take $x = M \in I$, and the above works. Assume then that $M + m = 2n - 1$ (therefore one must be n, while the other $n - 1$), when $\delta = 0$. It remains to settle the case when all given integers are taking only the values $-m$ or M, so $\pi^* M - \nu^* m = S$, with $\nu^* + \pi^* = N \geqslant 2n - 2$. Then $(2n-1)\nu^* = (\nu^* + \pi^*)M - S \geqslant (2n-2)M - (n-2)$, wherefore

[23] This type of reasoning is known from the Erdős problem, of finding a subsum divisible by n within any multiset of n integers.

$(2n-1)(\nu^*-M+1) \geqslant n+1-M > 0$, so $\nu^* \geqslant M$. Similarly, $(2n-1)\pi^* = (\nu^*+\pi^*)m+S \geqslant (2n-2)m+1$, wherefore $(2n-1)(\pi^*-m+1) \geqslant 2n-m > 0$, so $\pi^* \geqslant m$.

But then we can choose $m \leqslant \pi^*$ values M and $M \leqslant \nu^*$ values $-m$, in order to realize a zero sum $m \cdot M + M \cdot (-m) = 0$.

The results are best possible, since for $S = n-1$ we can exhibit a counterexample consisting of $\nu^* = n-1$ negatives equal to $-n+1$ and $\pi^* = n-1$ positives equal to n. No partial sum is zero, since $pn - q(n-1) = 0$ needs $n \mid q$, so $q \geqslant n$, while only $n-1$ such are available.

On the other hand, for $N = 2n-3$ we can exhibit a counterexample consisting of $\nu^* = n-2$ negatives equal to $-n$ and $\pi^* = n-1$ positives equal to $n-1$, when $S = 1$. No partial sum is zero, since $p(n-1) - qn = 0$ needs $n \mid p$, so $p \geqslant n$, while only $n-1$ such are available.

PROBLEMS AND SOLUTIONS

MATH STARS MATHEMATICAL COMPETITION

Problem 1. Show that for any positive integer n there exists a positive integer m such that

$$(1+\sqrt{2})^n = \sqrt{m} + \sqrt{m+1}.$$

Solution. Through simple induction it follows that there exist two sequences $(a_n)_{n\geqslant 1}$, $(b_n)_{n\geqslant 1}$ of positive integers such that

$$(1+\sqrt{2})^n = a_n + b_n\sqrt{2}.$$

Using conjugates we have $(1-\sqrt{2})^n = a_n - b_n\sqrt{2}$. Multiplying the two relations yields $(-1)^n = a_n^2 - 2b_n^2$. Hence the numbers a_n^2 and $2b_n^2$ are consecutive positive integers, but

$$(1+\sqrt{2})^n = \sqrt{a_n^2} + \sqrt{2b_n^2},$$

which concludes the solution.

Problem 2. Determine the positive integers n, x and y for which

$$2^x - n^{y+1} = \pm 1.$$

Solution. In both cases it is obvious n must be odd. Let us show first that $(n,x,y) = (3,3,1)$ is the only solution for the equation $2^x - n^{y+1} = -1$. Rewriting the equation as

$$2^x = n^{y+1} - 1 = (n-1)(n^y + n^{y-1} + \cdots + n + 1),$$

it follows y must be odd – otherwise $n^y + n^{y-1} + \cdots + n + 1 > 1$ would be odd, since $n \geqslant 1$ is odd. Hence

$$2^x = n^{y+1} - 1 = \left(n^{(y+1)/2} - 1\right)\left(n^{(y+1)/2} + 1\right).$$

Since $\left(n^{(y+1)/2} - 1, n^{(y+1)/2} + 1\right) = 2$, it follows that $n^{(y+1)/2} - 1 = 2$, whence $n = 3$ and $y = 1$, so $x = 3$. The numbers thus determined satisfy the equation.

Consider now the equation $2^x - n^{y+1} = 1$. We will show its solutions are $(n, x, y) = (1, 1, y)$, where $y \in \mathbb{N}^*$ is arbitrarily chosen. If $n = 1$, we immediately obtain the family of solutions in the above. Let then $n > 1$, n odd. If y is odd, then $y + 1$ is even, hence $n^{y+1} \equiv 1 \pmod 4$, since n is odd. Therefore, $2^x = n^{y+1} + 1 \equiv 2 \pmod 4$, i.e. $x = 1$, impossible for $n > 1$. If a y is even, then

$$2^x = n^{y+1} + 1 = (n+1)(n^y - n^{y-1} + \cdots - n + 1).$$

Since n is odd and y is even, the number $n^y - n^{y-1} + \cdots - n + 1$ is odd, so it must be equal to 1. Therefore, $n^{y+1} + 1 = n + 1$, impossible for $n > 1$ and $y > 0$.

Problem 3. Let ABC be a triangle and A_1, B_1, C_1 be the feet of the altitudes from A, B, C. Let A_2, respectively A_3, be the orthogonal projection of A_1 onto AB, respectively AC; points B_2, B_3 and C_2, C_3 are defined in an analogous way. The lines B_2B_3 and C_2C_3 meet at A_4, the lines C_2C_3 and A_2A_3 meet at B_4, while the lines A_2A_3 and B_2B_3 meet at C_4. Show that the lines AA_4, BB_4 and CC_4 are concurrent.

Solution. The gist of the problem is points $A_2, A_3, B_2, B_3, C_2, C_3$ are concyclic.

A first claim is that A_2A_3 is anti-parallel to BC, since quadrilateral $AA_2A_1A_3$ is cyclic, so $\angle AA_2A_3 = \angle AA_1A_3 = 90° - \angle A_3A_1C = \angle ACB$.

The next claim is that B_3C_2 is parallel to BC. Since quadrilateral $B_1C_1B_3C_2$ is cyclic, it follows $\angle AB_3C_2 = \angle AB_1C_1$, but $\angle AB_1C_1 = \angle ABC$ since the quadrilateral BCB_1C_1 is cyclic. Hence $\angle AB_3C_2 = \angle ABC$. Next we show that the quadrilateral $A_2A_3B_2C_3$ is cyclic, since A_3B_2 being parallel to AB, we prove as above. Hence $\angle B_2A_3C = \angle BAC$, and as A_2A_3 is anti-parallel to BC, it follows that $\angle AA_3A_2 = \angle ABC$. Therefore, $\angle A_2A_3B_2 = 180° - \angle AA_3A_2 - \angle B_2A_3C = 180° - \angle ABC - \angle BAC = \angle ACB$

Thus $\angle A_2A_3B_2 = \angle ACB$, and as A_2C_3 is parallel to BC (again, similarly with the above), it follows that $\angle A_2C_3B_2 = 180° - \angle ACB$, hence $\angle A_2A_3B_2 + \angle A_2C_3B_2 = 180°$, so the quadrilateral $A_2A_3B_2C_3$ is cyclic.

Next, we show the quadrilateral $C_2C_3B_2A_3$ is cyclic. Since C_2C_3 is anti-parallel to AB, it follows that $\angle CC_2C_3 = \angle ABC$. Since B_2A_3 is parallel to AB, it follows that $\angle CB_2A_3 = \angle ABC$. Hence $\angle CC_2C_3 = \angle CB_2A_3$, so the quadrilateral $C_2C_3B_2A_3$ is cyclic. From all of the above it follows that C_2 as well as A_2 lie on the circumcircle of triangle $\triangle C_3B_2A_2$, so points A_2, A_3, C_2, C_3, B_2 are concyclic. Similarly, we show that B_3 lies on this circle, so all six points are concyclic.

Let A' be the meeting point of lines A_2A_3 and BC, B' be the meeting point of lines B_2B_3 and AC, and C' be the meeting point of lines C_2C_3 and AB. We claim A', B', C' are collinear. Let us apply Pascal's Theorem for the six points $A_2, A_3, C_2, C_3, B_2, B_3$. Since $A_2A_3 \cap C_3B_2 = \{A'\}$, $A_3C_2 \cap B_2B_3 = \{B'\}$, $C_2C_3 \cap B_3A_2 = \{C'\}$, it follows that the claim is valid.

But from the fact that A', B', C' are collinear it follows that triangles $\triangle ABC$ and $\triangle A_4B_4C_4$ are perspective, since $BC \cap B_4C_4 = \{A'\}$, $AC \cap A_4C_4 = \{B'\}$, $AB \cap A_4B_4 = \{C'\}$. Applying Desargues' Theorem yields the fact that lines AA_4, BB_4 and CC_4 are concurrent, closing the proof.

Problem 4. At a table-tennis tournament, the $n \geqslant 2$ participants play, each against each, exactly one match. Show that exactly one of the following two situations occurs at the end of the tournament:

(1) the n participants can be labeled with the numbers $1, 2, \ldots, n$ such that 1 beat 2, 2 beat 3, and so on, $n - 1$ beat n and n beat 1;

(2) the n participants can be partitioned in two non-empty sets A, B, such that every member of A beat each member of B.

Solution. (D. Schwarz) In graph-theoretical terminology, let G be the complete digraph of the participants, with edges orientated from winners toward losers. If there exists a player who won no match, i.e. was beaten by all other (vertex of out-degree zero), then we can take as B that player (and A the rest of the players), to fulfill case (2). Otherwise, we can start from an arbitrary vertex an oriented path which, due to G being finite, must pass again through some vertex, thus creating a cycle. Let Γ be a maximal length cycle. If it contains all vertices, the case (1) occurs, through this Hamiltonian cycle. If not, and x is a vertex exterior to $\Gamma = c_1c_2\dots c_mc_1$, consider the orientation of the edge xc_1. We must have xc_m, otherwise we can build the cycle $c_1c_2\dots c_mxc_1$, longer than Γ. Similarly, we must have xc_i for all $i = 1, 2, \dots, m$. If the orientation of the edge is c_1x, a similar reasoning yields c_ix, for all $i = 1, 2, \dots m$. Let X be the set of vertices of the first kind, and Y that of the vertices of the second kind; X and Y cannot be simultaneously empty. If $X = \emptyset$, we can take $A = \Gamma$, $B = Y$. If $Y = \emptyset$, we can take $A = X$ and $B = \Gamma$. If X and Y are both non-empty, it follows that for any $x \in X$ and $y \in Y$ we have the orientation of the edge xy, otherwise we build the cycle $xc_1\dots c_myx$, longer than Γ. Then we can take $A = X$, $B = \Gamma \cup Y$ (or $A = X \cup \Gamma$, $B = Y$). Obviously, situations (1) and (2) are mutually exclusive, since a graph satisfying property (2) cannot contain a Hamiltonian cycle, as no edge is oriented from B toward A.

Problem 5. Show there uniquely exists a function $f : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ which simultaneously satisfies the following three conditions:

(1) $f(x,y) = f(y,x)$, for all $x, y \in \mathbb{N}^*$;

(2) $f(x,x) = x$, for all $x \in \mathbb{N}^*$; and

(3) $(y-x)f(x,y) = yf(x, y-x)$, for all $x, y \in \mathbb{N}^*$, $y > x$.

Solution. The function $f(x,y) = \operatorname{lcm}(x,y)$ fulfills the conditions from the statement. To prove its uniqueness, let us assume the existence of two such functions, f and g, $f \neq g$. From all pairs (x,y) for which $f(x,y) \neq g(x,y)$, let us choose one (x_0, y_0) such that the product x_0y_0 be minimal. Condition (2) implies $x_0 \neq y_0$, while condition (1) allows us to suppose in the sequel that $x_0 < y_0$. Since $x_0(y_0 - x_0) < x_0y_0$, the minimality condition implies $f(x_0, y_0 - x_0) = g(x_0, y_0 - x_0)$. Due to (3)

$$(y_0 - x_0)f(x_0,y_0) = y_0f(x_0, y_0 - x_0) = y_0g(x_0, y_0 - x_0) = (y_0 - x_0)g(x_0,y_0),$$

i.e. $f(x_0,y_0) = g(x_0,y_0)$, contradiction.

Another way would be to notice that, through induction on $x + y$, the values of the function f are uniquely determined to be $f(x,y) = \operatorname{lcm}(x,y)$.

Problem 6. Let have $n > 3$ points in the space, four by four non-coplanar, any two of them connected by wires.

(1) By cutting the $n-1$ wires that connect one point from the others, that point is disconnected (becomes *isolated*). Show that cutting less than $n-1$ wires does not disconnect the structure.

(2) Determine the minimum number of wires needed to be cut, in order to disconnect the structure, with no point becoming isolated.

Solution. (D. Schwarz) The complete graph K_n of order n has $\binom{n}{2}$ edges. Let N be a number of edges which, by being removed, disconnect K_n, and let m be the least of the orders of the connected components thusly obtained. Then $n(n-1)/2 - N \leqslant m(m-1)/2 + (n-m)(n-m-1)/2$, hence $N \geqslant m(n-m)$. Since $m \geqslant 1$, it follows that $N \geqslant n-1$, which solves part (1) of the problem. If, moreover, $m \geqslant 2$, then $N \geqslant 2n-4$. An example realizing exactly this value is given by the $(n-2) + (n-2)$ edges which connect two of the points with the other $n-2$. This solves part (2) of the problem.[24]

Problem 7. Let $A_0 \cdots A_{n-1}$ be a regular n-gon. For each index i, consider a point B_i lying on the side A_iA_{i+1}, such that $A_iB_i < \frac{1}{2}A_iA_{i+1}$, and a point C_i lying on the segment A_iB_i (indices are reduced modulo n). Show that the perimeter of the polygon $C_0 \cdots C_{n-1}$ is at least as large as the perimeter of the polygon $B_0 \cdots B_{n-1}$.

Solution. The conclusion follows by repeated application of the following

Lemma. *Let $ABCD$ be a convex polygonal line (points A and D are situated on the same side of line BC), such that $AB = CD$ and $\angle ABC = \angle BCD$. Let K, M and N be points lying on the segments AB, BC and CD, such that*

[24] All follows from the fact that the function $m \mapsto m(n-m)$ is non-decreasing for $1 \leqslant m \leqslant \lfloor n/2 \rfloor$, which is exactly the variation interval for m, since there will be at least two connected components.

$AK < \frac{1}{2}AB$, $BM < \frac{1}{2}BC$ and $CN < \frac{1}{2}CD$. If L lies on the segment BM, then $KL + LN \geqslant KM + MN$.

Proof. To prove the lemma, consider N', the symmetrical of point N with respect to the line BC. The segments BC and KN' will meet at a point P, such that $CP < \frac{1}{2}BC$; the inequality follows from the fact that the similitude ratio of the triangles BKP and $CN'P$ is larger than 1. Therefore, point M lies in the interior of the triangle KLN', hence $KL + LN = KL + LN' \geqslant KM + MN' = KM + MN$. □

Denote by $\mathcal{P}(\Gamma)$ the perimeter of a polygon Γ. A first application of the lemma, to the polygonal line $A_{n-1}A_0A_1A_2$ ($K = C_{n-1}$, $L = C_0$, $M = B_0$, $N = C_1$), shows that

$$\mathcal{P}(C_0C_1 \cdots C_{n-1}) \geqslant \mathcal{P}(B_0C_1 \cdots C_{n-1}).$$

Successive applications of the lemma yield

$$\begin{array}{rcll} \mathcal{P}(C_0 \cdots C_{n-1}) & \geqslant & \mathcal{P}(B_0C_1 \cdots C_{n-1}) & \geqslant \cdots \\ & \geqslant & \mathcal{P}(B_0 \cdots B_iC_{i+1} \cdots C_{n-1}) & \geqslant \cdots \\ & \geqslant & \mathcal{P}(B_0 \cdots B_{n-1}). & \end{array}$$

Problem 8. Prove that any set of 27 positive integers, ranging between 1 and 2007, contains three distinct elements a, b, c such that $\gcd(a, b)$ (the greatest common divisor of a and b) divides c.

Solution. It is rather more convenient to consider the converse problem:

Find a universal lower bound f(m) for the largest element $\max(A)$ *of a set A of m positive integers, $m \geqslant 3$, with the property that* $\gcd(a, b)$ *does not divide c for any distinct $a, b, c \in A$.*

We will call such a set *proper*. In what follows, all elements denoted by different letters are to be considered distinct. Let us denote by M_m the minimal value of the largest element $\max(A)$ of a proper set A with m elements.[25] It follows

[25] The existence of M_m is guaranteed by the following example: consider $p_1, p_2, \ldots, p_m$ to be the first m primes, and take $c_i = \prod_{j \neq i}^{m} p_j$; the set $C_m = \{c_i \ ; \ 1 \leqslant i \leqslant m\}$ will be called *canonical*, is obviously proper, and its largest element is c_1, hence $M_m \leqslant \prod_{j=2}^{m} p_j$.

that any set A with m elements, all less than $f(m)$, will not be proper, hence for $f(m) > 2007$ will be eligible.

The key idea to our solution is to consider the set $\{\gcd(x, y) \ ; \ x, y \in A\}$. A being proper is then equivalent to the fact that no two such gcd's are dividing one another; in particular, no two are equal, so we have $\binom{m}{2}$ distinct values. Now, if we have an upper bound $\gcd(x, y) \leqslant N$ for all $x, y \in A$, then we need have $\binom{m}{2} \leqslant (N + 1)/2$, otherwise a famous Erdös result states that there will exist a pair of gcd's, one dividing the other.[26]

But we cannot have $x = \gcd(x, y)$, as then, for a third element z, $\gcd(x, z)$ divides x, and thus a fortiori y (and similarly, nor $y = \gcd(x, y)$), therefore $\max(x/\gcd(x, y), y/\gcd(x, y)) \geqslant 3$, hence $\gcd(x, y) \leqslant \max(x, y)/3 \leqslant M_m/3$. The above result then provides $M_m \geqslant 3(m^2 - m - 1)$. For $m = 27$ this yields $M_{27} \geqslant 2103 > 2007$, so the statement of the problem is certainly true for the value 27.

OPEN QUESTION. Improve this result, by lowering the number 27 necessary to obtain the stated property.

Partial solution. Refining our reasoning, we will prove that $M_m \geqslant 15(m^2 - m - 1) = f(m)$ for $m \geqslant 7$; then, as $f(13) = 2325 > 2007$, the statement of the problem remains true for 13 instead of 27 (but $f(12) = 1965 < 2007$, so we cannot lower the value 13 to 12).

For $x, y \in A$, $x/\gcd(x, y)$ and $y/\gcd(x, y)$ are co-prime, and none is equal to 1. If at least one of them is not less than 15 (and this for all $x, y \in A$), then $\gcd(x, y) \leqslant \max(x, y)/15 \leqslant M_m/15$, and similar reasoning as above yields $M_m \geqslant 15(m^2 - m - 1) = f(m)$. If, on the other hand, there is a pair such that both are less than 15, as at least one (wlog be it $y/\gcd(x, y)$) will be odd, it will be 3, 5, 7, $9 = 3^2$, 11 or 13, thus p^e with p odd prime, and $y = p^e \gcd(x, y)$. But then, for any third element z, we need have p dividing z (else $\gcd(y, z)$ would divide $\gcd(x, y)$). We thus obtain a proper set $B = \{t/p; \ t \in A, \ t \neq x\}$ of

[26] A sketchy well-known proof (for self-containment): as there are exactly $\lfloor (N + 1)/2 \rfloor$ odd positive integers (not larger than N), any set of more than $(N + 1)/2$ positive integers (not larger than N) will contain (PH principle) two that share the same maximal odd factor; therefore, one of them must divide the other.

$m-1$ elements, and therefore $M_m \geqslant pM_{m-1} \geqslant 3M_{m-1}$. In conclusion, $M_m \geqslant \min(f(m), 3M_{m-1})$.

Now, $f(m) < 3f(m-1)$ for $m \geqslant 4$, so by simple induction: $M_4 \geqslant \min(f(4), 3M_3)$, and using $M_k \geqslant \min(f(k), 3^{k-3}M_3)$ for $4 \leqslant k < m$ as induction hypothesis, we get

$$M_m \geqslant \min(f(m), 3M_{m-1}) \geqslant \min(f(m), 3\min(f(m-1), 3^{(m-1)-3}M_3))$$
$$= \min(f(m), 3f(m-1), 3^{m-3}M_3) = \min(f(m), 3^{m-3}M_3).$$

As $M_m \geqslant 3(m^2-m-1)$ (a result obtained in the above), we get, for $m = 3$, $M_3 \geqslant 15$, but $M_3 \leqslant 15$ (for the canonical set C_3), hence $M_3 = 15$. Simple calculations and induction show that $f(m) < 3^{m-3}15 = 3^{m-3}M_3$ for $m \geqslant 7$, hence $M_m \geqslant 15(m^2-m-1) = f(m)$ for $m \geqslant 7$.

REMARKS. Reversing the formula for $f(m)$, with elements bounded by M, we get $m > \sqrt{M/15}+1$, which under the conditions of the problem yields $m > \sqrt{2007/15}+1$, that is $m \geqslant 13$.

As the canonical set C_5 has its largest element equal to $1155 < 2007$, it follows $M_5 < 2007$; the author strongly suspects that the true threshold value for the statement of the problem is 6 (instead of 13), but he is unable to prove it for the time being. A different approach, by M. Dumitrescu, comes nearer and may even positively solve this.

However, the conjecture $M_m = \prod_{j=2}^{m} p_j$, which is definitely true for $m = 3$ and quite probably also true for $m = 4, 5$ or 6, is in general false for $m \geqslant 7$, as the following example shows:

Take $a_i = \prod_{j\neq i}^{m-1} p_j{}^{e_j}$, $1 \leqslant i \leqslant m-1$, $e_1 = e_2 = e_3 = 2$ (with the remaining ones equal to 1), and $a_m = \prod_{j=1}^{m-1} p_j$; this set is proper, and its largest element is $3^2 5^2 7 \cdots p_{m-1} < p_2 p_3 p_4 \cdots p_m$, as $15 < p_m$ for $m \geqslant 7$ ($p_7 = 17 > 15$).

PROBLEMS AND SOLUTIONS

CLOCK-TOWER SCHOOL SENIORS COMPETITION

Problem 1. Prove that, for any $n \in \mathbb{N}$, $n \geqslant 2$, the Diophantine equation

$$1 + x_1^2 + \cdots + x_n^2 = y^2$$

has infinitely many positive integer solutions with $1 < x_1 < \cdots < x_n$.

Solution. Clearly the family

$$x_1 = 2k, \quad x_2 = 2k^2, \quad y = 2k^2+1, \quad k \geqslant 2 \quad (\text{so } 1 < x_1 < x_2 < y),$$

yields infinitely many such solutions for $n = 2$. Moreover, y turns to be odd.

But any odd number $y = 2t+1 = (t+1)^2 - t^2$ is part of a Pythagorean triplet $(y, z = 2t(t+1), w = (t+1)^2 + t^2 = 2(t^2+t)+1)$

$$((t+1)^2 - t^2)^2 + (2t(t+1))^2 = ((t+1)^2 + t^2)^2,$$

with w again odd.

With a little bit of induction now, for any family of solutions (for n) having odd y, one gets a family of solutions (for $n+1$) having odd y, through the simple expedient of taking $x_{n+1} = z$ and $y = w$.

Alternative solution. Clearly one can choose (infinitely often) integers $1 < x_1 < \cdots < x_{n-1}$ such that

$$x_1^2 + \cdots + x_{n-1}^2 = 2k, \quad k > 2,$$

then, by taking $x_n = k > \sqrt{2k} \geqslant x_{n-1}$ one gets

$$1 + x_1^2 + \cdots + x_{n-1}^2 + x_n^2 = 1 + 2k + k^2 = (k+1)^2,$$

thereby fulfilling the requirement.

Problem 2. Let ABC be an acute-angled triangle, and ω, respectively Ω, be its incircle and circumcircle. Circle ω_A is tangent (internal) to Ω at A, and tangent (external) to ω at A_1. Points B_1 and C_1 are similarly obtained, starting with B, respectively C. Prove that lines AA_1, BB_1 and CC_1 are concurrent.

Solution. Denote by $H_{\Omega\omega}$ the homothety transforming ω into Ω (of center X), by $H_{\omega_A\omega}$ the homothety transforming ω into ω_A (obviously of center A_1), and by $H_{\Omega\omega_A}$ the homothety transforming ω_A into Ω (obviously of center A). Since $H_{\Omega\omega} = H_{\Omega\omega_A} \circ H_{\omega_A\omega}$, it follows X, A and A_1 are collinear.

Problem 3. In the Cartesian coordinate plane define the strips

$$S_n := \{(x,y);\ n \leqslant x < n+1\},$$

for every integer n. Assume each strip is colored either white or black. Prove one can place any rectangle R, not a square, in the plane, such that its vertices share a same color.

Solution. For easy reference, label R's vertices A, B, C, D. If the coloring is monochromatic, the result is trivial, so assume the contrary. If the rectangle has (at least) a side of non-integer length s, be it AB, place the rectangle with side AD lying on a boundary line between white and black strips, and B to the right of A. Then side BC will fall into a strip which, if colored the same as AD, yields the result, while if colored the opposite, allows slightly sliding the rectangle to the left, again yielding the result.

On the other hand, for a rectangle with integer length side s, the sliding method described in the above fails, for **just** those unfit colorings precisely described by Lemma 1 in the sequel.

When both sides p, q are integer, denote $(p,q) = d$, $p = da$, $q = db$. Since R is not a square, p, q are distinct, e.g. $p > q$. Lemma 2 shows that the only unfit colorings for the sliding method described in the above have periodicity $2d$. Moreover, we need have both a, b odd, so $a \geqslant 3$. Wlog assume $AB = p$.

Place the rectangle with A on a boundary line between white and black strips, B on the boundary line situated $2d$ to the right, and D to the left and down of A. This is possible, since $AB = p = ad \geqslant 3d > 2d$. Then A, B share the same color, and C, D also share the same color, since the horizontal distance between them is also $2d$. If A, D share the same color, we are done, so assume the contrary. Denote X, Y the projections of B, respectively D, onto the boundary line through A, so $BX = 2d$. Triangles ΔAXB and ΔDYA are similar, so $DY^2AB^2 = AX^2AD^2$. Then $DY^2p^2 = (p^2 - 4d^2)q^2$, so $DY = \frac{b}{a}d\sqrt{a^2-4}$. But $\sqrt{a^2-4}$ is irrational, hence $DY \notin \mathbb{Z}$. Then slightly sliding R to the left yields a monochrome coloring of its vertices.

Let us present the results of the lemmas in free-monoid terminology. The alphabet A is the set $\{w, b\}$ of colors, and the finite words are colorings of finite, contiguous groups of strips. A coloring of (all) the strips may be construed as being an infinite word C on the alphabet, with the (strip) position indexed on $\mathbb{Z}$ (the index 0 is attributed to strip S_0), and the color of the strip of index i being given by $\mathcal{C}(i)$. For a finite word V, denote by $|V|$ the length of V, and V^* the infinite word of period V.

Let us define the mapping φ such that $\varphi(V)$ is the word obtained from V by replacing each color in it by its opposite (the extension to words of $\varphi(w) = b$, $\varphi(b) = w$). Clearly, $\varphi^2 = \varphi \circ \varphi = \text{id}$.

LEMMA 1. *The pattern of an unfit coloring C for an integer side s is periodic, $C = (S\varphi(S))^*$, with S an arbitrary word with $|S| = s$.*

The sliding method only fails if at any moment the colors of the strips situated s apart are the opposite; symbolically this is represented by the formula $\mathcal{C}(i+s) = \varphi(\mathcal{C}(i))$ for all $i \in \mathbb{Z}$. In turn, this yields the claimed result. Moreover, through simple induction, one gets that $\mathcal{C}(i+ks) = \varphi^k(\mathcal{C}(i))$, for all $i \in \mathbb{Z}$ and all $k \in \mathbb{Z}$. □

LEMMA 2. *If the coloring C is unfit for both integer sides p, q, then its pattern is periodic, $C = (S\varphi(S))^*$, with S an arbitrary word with $|S| = d = (p,q)$. Moreover, both $\frac{p}{d}$, $\frac{q}{d}$ need be odd.*

One has $d = up + vq$, for some $u, v \in \mathbb{Z}$. Now, according to Lemma 1, one has $\mathcal{C}(i + kd) = \mathcal{C}((i + kup) + kvq) = \varphi^{kv}(\mathcal{C}(i + kup)) = \varphi^{k(u+v)}(\mathcal{C}(i))$, for all $i \in \mathbb{Z}$ and all $k \in \mathbb{Z}$.

The coloring being unfit for side p, we have, according to Lemma 1, $\mathcal{C}(p) = \varphi(\mathcal{C}(0))$. On the other hand, $\mathcal{C}(p) = \mathcal{C}(\frac{p}{d}d) = \varphi^{\frac{p}{d}(u+v)}(\mathcal{C}(0))$. Then $1 \equiv \frac{p}{d}(u + v) \pmod 2$, hence $\frac{p}{d}$ must be odd. Similarly, $\frac{q}{d}$ must be odd. Also $u + v$ must be odd, so $\mathcal{C}(i + kd) = \varphi^k(\mathcal{C}(i))$, for all $i \in \mathbb{Z}$ and all $k \in \mathbb{Z}$. □

REMARKS. Clearly, squares of integer side s cannot be always placed, since for a coloring made of alternating colored groups of s contiguous strips, the square will always have a white vertex, and always a black vertex as well. Notice that Lemma 2 also applies for equal integer sides.

Problem 4. Let $(a_n)_{n\geqslant 0}$ be a real sequence having

$$a_{n+1} + a_{n-1} = |a_n|, \quad \text{for all } n \geqslant 1.$$

Prove the sequence is periodic.

Solution. (D. Schwarz) Since $a_{n+1} + a_{n-1} = |a_n| \geqslant 0$, the sequence contains infinitely many non-negative terms. Moreover, among any four consecutive terms, at least one needs be non-negative, and one non-positive.

Starting with two consecutive terms (among the first 8), the first, $-x$, non-positive and the latter, y, non-negative, by following the recurrence relation one gets periodicity of length 9:

$-x, y, x + y, x, -y, y - x$, (then, for $y - x \geqslant 0$), $2y - x, y, x - y, -x, y, \ldots$

or

$-x, y, x + y, x, -y, y - x$, (then, for $y - x \leqslant 0$), $x, 2x - y, x - y, -x, y, \ldots$.

Conversely, starting in the opposite direction (among terms of index greater than 100, let's say), since the recurrence is symmetrical, one gets to the same conclusion. Therefore, the mid-part will be periodical of two periods of length 9, only possible if the periods coincide.

Problem 5. A rectangle D is partitioned in (more than one) rectangles having their sides parallel to those of rectangle D. It is given that any line parallel to one of the sides of D, and having common points with the interior of D, will also have common points with the interior of (at least) one of the rectangles in the partition. Prove that in this partition there is (at least) a rectangle that has **no** common points with the border of D.

Solution. Any solution must carefully do the 'analysis situ' of partition rectangles. The proof goes by contradiction – assume all partition rectangles are attached to side(s) of D.

First, it should be clear that there must then be *opposite* (i.e. attached to opposite sides of D) rectangles having a common point. Now, by studying the (few) possible cases, one gets that avoiding a *splitting* line (i.e. crossing D without having common points with the interior of any rectangle in the partition, just with borders), is impossible.

Problem 6. Given an odd integer $n > 3$ not divisible by 3, show that there exist distinct odd, positive integers a, b, and c such that

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Solution. Since n is odd and is not divisible by 3, either $n \equiv 1 \pmod 6$ or $n \equiv 5 \pmod 6$.

In the first case, $n = 6m + 1$ for some positive integer m, and the numbers $a = 2m + 1$, $b = (2m + 1)(4m + 1)$, and $c = (4m + 1)(6m + 1)$ do the job.

In the second case, $n \equiv 2 \pmod 3$, so there exists an integer $m \geqslant 1$ such that

$$n \equiv 2 + 2 \cdot 3 + \cdots + 2 \cdot 3^{m-1} + \alpha \cdot 3^m \pmod{3^{m+1}}$$

with $\alpha \in \{0, 1\}$.

If $\alpha = 0$, then $(3^m + 1)n = 3^{m+1}a - 1$ for some odd, positive integer a, and

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{3^m a} + \frac{1}{3^m an}$$

is the desired decomposition.

If $\alpha = 1$, then $3^m + n + 1 = 3^{m+1}k$ for some odd, positive integer k, and the numbers $a = 3^m k$, $b = 3^m kn$, and $c = kn$ do the job.

PROBLEMS AND SOLUTIONS

CLOCK-TOWER SCHOOL JUNIORS COMPETITION

Problem 1. Consider a circle of center O and a chord AB of it (not a diameter). Take a point T on the ray OB. The perpendicular at T onto OB meets the chord AB at C and the circle at D and E. Denote by S the orthogonal projection of T onto the chord AB. Show that $AS \cdot BC = TE \cdot TD$.

Solution. We have $TE = TD$. Then $CA \cdot CB = CD \cdot CE$ (from the power-of-a-point law, or from the similitude of triangles ACD and BEC). Moreover, $TC^2 = CS \cdot CB$ (known result in a right-angled triangle).

But then

$$\begin{aligned} AS \cdot BC &= (AC + CS) \cdot BC = CD \cdot CE + CT^2 \\ &= (TD - TC)(TD + TC) + TC^2 = TD^2 = TE \cdot TD. \end{aligned}$$

Problem 2. The last digit in the decimal representation of number a^2+ab+b^2, with $a, b \in \mathbb{N}^*$, is 0. Find its second-to-last digit.

Solution. Clearly a, b are even, so 4 divides $a^2 + ab + b^2$. On the other hand, 5 divides $a^2 + ab + b^2$, so 5 divides $a^3 - b^3$. Therefore $a \equiv b \pmod 5$, hence $5 \mid 3a^2$. Consequently, $5 \mid a$ and $25 \mid a^2 + ab + b^2$. Then $100 \mid a^2 + ab + b^2$, i.e. the second-to-last digit is 0.

Alternative solution. The last digits of cubes of integers make a periodical sequence of period 10, hence $a \equiv b \pmod{10}$, whence 10 divides $3a^2$.

Problem 3. Partition a triangle into (smaller) triangles. Show that the sum of the lengths of the lesser altitudes of the triangles of the partition is at least equal to the length of the lesser altitude of the given triangle.

Solution. Let $h_1, \dots, h_n$ be the lesser altitudes of the triangles of the partition, and let h be the lesser altitude of the given triangle. Let $a_1, \dots, a_n$, respectively a, be the corresponding sides. Then $\sum a_i \cdot h_i = a \cdot h$. Since $a_i \leqslant a$, for all $1 \leqslant i \leqslant n$, it follows

$$a \cdot \sum h_i \geqslant a \cdot h,$$

whence the required result.

Clearly equality is only possible when the partition is made of one triangle only – the given one!

REMARKS. There is a famous result worth mentioning, from which this problem flows as an immediate corollary. Call *strip* of breadth b the closed part of the plane consisting of all points that lie between two parallel lines at distance b from each other, and call *oval* a bounded, closed convex set in the plane. The result is known as

The Plank Problem. *If an oval can be covered by n strips of breadths $b_1, b_2, \dots, b_n$, then it can also be covered by a single strip of breadth $b = b_1+b_2+\cdots+b_n$.*

This has been conjectured in 1932 by A. Tarski, and established by T. Bang in 1951 (quote from H. Hadwiger & H. Debrunner – *Combinatorial Geometry in the Plane*).

Now, the strips determined by the sides a_i (corresponding to the lesser altitudes) and the parallels to them through the opposite vertex of the triangles in the partition have breadths h_i, and clearly cover the given triangle. On the other hand, the least breadth of one strip that covers the given triangle must be h.

Problem 4. Consider any 25 points, three by three non-collinear, in the interior of a square of side length 3. Show that there exist four among them that form a quadrilateral perimeter less than 5.

Solution. Partition the square into 6 rectangles 1×1.5. Applying the pigeonhole principle (Dirichlet principle), since $25 = 4{\cdot}6{+}1$, at least one of the rectangles must contain at least five of the given points.

But then four among those must be the vertices of a convex quadrilateral, of lesser perimeter than that of the 1×1.5 rectangle, which amounts to 5. The only case when these perimeters are equal is when the four points coincide with the vertices of the rectangle, but then some three of them, together with the fifth one determine a convex quadrilateral of lesser perimeter.

REMARKS. Some contestants partitioned the square into 8 rectangles 0.75×1.5. Applying the pigeon-hole principle, since $25 = 3 \cdot 8 + 1$, at least one of the rectangles must contain at least four of the given points. They now concluded that the perimeter of a quadrilateral determined by these points must be at most that of the 0.75×1.5 rectangle, which amounts to $4.5 < 5$. Alternatively, since $19 = 3 \cdot 6 + 1$, this strengthens the problem by showing that 19 points were sufficient to reach the conclusion.

Of course this is insofar unfounded, as those four points may not be in convex position – and then three concave quadrilaterals have to be considered, while clearly a non-convex figure contained in a rectangle may have a larger perimeter than that of the rectangle. However, for quadrilaterals, it may be shown that at least one of the three concave quadrilaterals that may be considered is necessarily of a lesser perimeter. The proof is elementary, though extremely elegant, and is left to the readers to enjoy discovering!

Problem 5. A positive integer has, in its decimal representation, 2008 digits equal to 1, 2008 digits equal to 4, while the rest of its digits are equal to 0. Show that this number cannot be a perfect square.

Solution. The sum $s(n)$ of the digits of n is of the form $\mathcal{M}3 + 2$. Since 3 divides $n - s(n)$, then n also must be of the form $\mathcal{M}3 + 2$, so cannot be a perfect square.

Problem 6. Let $\mathcal{P}$ be the set of all points of the plane, and $O \in \mathcal{P}$ fixed. The function $f : \mathcal{P} \setminus \{O\} \to \mathbb{R}$ has the property:

For any four distinct points $A, B, C, D \in \mathcal{P} \setminus \{O\}$ with $\Delta AOB \sim \Delta COD$,

$$f(A) - f(B) + f(C) - f(D) = 0 \textit{ occurs.}$$

Prove the function f is constant.

Solution. Let $X, Y \in \mathcal{P} \setminus \{O\}$ be two arbitrary distinct points. The circle centered in O and of radius OX meets the line OY at A, such that points A, O, Y are collinear in this order, while the circle centered in O and of radius OY meets the line OX at B, such that points X, O, B are collinear in this order.

Since triangles AOX and BOY are isosceles of apex O, and $\angle AOX = \angle BOY$, we have $\Delta AOX \sim \Delta BOY$, but also $\Delta AOX \sim \Delta YOB$, so relations $f(A) - f(X) + f(B) - f(Y) = 0$, and $f(A) - f(X) + f(Y) - f(B) = 0$ occur.

Summing them yields $f(A) = f(X)$ and $f(B) = f(Y)$. On the other hand, since triangles AOX and BOY are isosceles, it follows that triangles AOB and XOY are congruent, so $f(A) - f(B) + f(X) - f(Y) = 0$, whence $f(X) = f(Y)$.

Problem 7. For any real numbers $a, b, c > 0$, with $abc = 8$, prove

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leqslant 0.$$

Solution.

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \leqslant 0 \Leftrightarrow 3 - 3\sum \frac{1}{a+1} \leqslant 0 \Leftrightarrow 1 \leqslant \sum \frac{1}{a+1}.$$

We can take $a = 2\frac{x}{y}, b = 2\frac{y}{z}, c = 2\frac{z}{x}$ (a known *trick*), to have

$$\sum \frac{1}{a+1} = \sum \frac{y^2}{2xy + y^2} \geqslant \frac{(x+y+z)^2}{x^2 + y^2 + z^2 + 2xy + 2yz + 2zx} = 1.$$

Problem 8. Let p be a prime, and q an integer, not divisible by p. Prove there exist infinitely many integers k such that pq divides $q^k + 1 - k$.

Solution. Clearly q must divide $1 - k$, so let's take $k = 1 + qs$, with $s \in \mathbb{N}$. Then $q^k + 1 - k = q(q^{qs} - s)$, so we are left with finding values for s with $p \mid q^{qs} - s$.

If $s \equiv 1 \pmod p$, then it comes to $(q^q)^s \equiv 1 \pmod p$.

An example for a *good* s is obtained applying Fermat's Theorem – simply take $s = (p-1)^n$, since we have $(p, q^q) = 1$. Any even value for n will satisfy the requirement $s \equiv 1 \pmod p$.

Therefore, $s = (p-1)^{2t}$, with arbitrary $t \in \mathbb{N}$, will do.

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