Mircea Becheanu
Marian Andronache
Mihai Bălună
Radu Gologan
Dinu Șerbănescu
Valentin Vornicu

R.M.C.

2003

ROMANIAN MATHEMATICAL COMPETITIONS

Societatea de Științe Matematice din România
Theta Foundation

UNIVERSITY OF BUCHAREST
FACULTY OF MATHEMATICS and
COMPUTER SCIENCE

Prof. MIRCEA BECHEANU, Ph.D.
Vice President of Romanian Mathematical Society

14 Academiei Street
70109 Bucharest
Romania

Tel.&Fax:++40 21 3124072
Mobile:++40 744904343
Email: bechmir@rms.unibuc.ro

Mircea Becheanu
Marian Andronache
Mihai Bălună
Radu Gologan
Dinu Şerbănescu
Valentin Vornicu

R.M.C.

2003

ROMANIAN MATHEMATICAL COMPETITIONS

Societatea de Ştiinţe Matematice din România
Theta Foundation

ROMANIAN MATHEMATICAL COMPETITIONS

Distributed by:
Societatea de Ştiinţe Matematice din România
Str. Academiei 14, 70103 Bucureşti, România
E-mail: office@rms.unibuc.ro
Home page: http://www.ssmro.com
tel/fax: +40(21)3124072

Typed and technically edited by Theta
http://www.theta.ro

Printed at NECOZ PREST
Str. Oracolului 8, Bucureşti, tel.: +40(21)6106328

ISBN 973–0–03138–X

PREFACE

It becomes a tradition to publish an English version of the most interesting problems given at the Romanian Mathematical Olympiad and other Romanian Mathematical Competitions in a booklet form.

We present the nineth edition of this collection, realized in cooperation by the Romanian Mathematical Society and the Theta Foundation.

We would like to thank some of the sponsors of the Romanian Mathematical Olympiad and of the Romanian Team for the International Mathematical Olympiad and the Balkan Competitions. These thanks are due to "Winsoft", "The Theta Foundation", "The Gill Publishing House" and "The Sigma Foundation".

Thanks are also due to Luminiţa Stafi and Barbara Ionescu, from the The Theta Foundation, for their excellent typing.

Bucharest
July 1st, 2003

Mircea Becheanu
Radu Gologan

CONTENTS

Part I. THE 54^{th} NATIONAL MATHEMATICAL OLYMPIAD

PROPOSED PROBLEMS

I.1. FIRST ROUND — CITY OF BUCHAREST

March 26, 2003

9^{th} GRADE

PROBLEM 1. Find the integer part of the number

$$\left.\sqrt[3]{24+\sqrt[3]{24+\cdots+\sqrt[3]{24}}}\right\} n \text{ roots},$$

where $n \geqslant 1$.

* * *

PROBLEM 2. Let x and y be real numbers so that x^2+y^2, x^3+y^3 and x^4+y^4 are rational numbers. Prove that $x+y$ and xy are also rationals.

Sorin Rădulescu, Costel Chiteş

PROBLEM 3. Find the function $f : \mathbf{R} \to \mathbf{R}$ with the properties:
(a) $f(x) = x^2$ for all $x \in [0,1)$;
(b) $f(x+1) = f(x) + x$ for all $x \in \mathbf{R}$.

Laurenţiu Panaitopol

PROBLEM 4. Consider the points A, B, C, D in a plane, not three of them collinear. Points H_1 and H_2 are the orthocenters of the triangles ABC and ABD, respectively.

Prove that A, B, C, D lie on the same circle if and only if

$$\overrightarrow{H_1H_2} = \overrightarrow{CD}.$$

Marian Andronache

PROBLEM 5. An arbitrary point M is considered on the side BC of the triangle ABC. Let $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ be the incircles of the triangles ABC, ABM, ACM, respectively.

(a) Prove that $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent if and only if $M \in \mathcal{C}_0$.

(b) Suppose $M \in \mathcal{C}_0$ and let S and D be the midpoints of the segments AM and BC. Prove that $p\overrightarrow{AI} = a\overrightarrow{AS} + (p-a)\overrightarrow{AD}$. Furthermore, derive that points I, S, D are collinear and $\frac{IS}{ID} = \frac{p-a}{a}$.

Virgil Nicula

10th GRADE

PROBLEM 1. Let $ABCD$ be a convex quadrilateral and let M be the midpoint of the side CD. Lines BM and AM are perpendicular and $AB = BC + AD$.

Prove that lines BC and AD are parallel.

Laurenţiu Panaitopol

PROBLEM 2. Let a, b, c, d be complex numbers with equal absolute values such that $a + b + c = d$.

Prove that d is equal to one of the numbers a, b or c.

Marcel Ţena

PROBLEM 3. Let n and p be positive integers with $p > 2^n$. Prove that the integer part of the number $\sum\limits_{k=0}^{n} \sqrt[p]{1 + \binom{n}{k}}$ is equal to $n + 1$.

Virgil Nicula

PROBLEM 4. Prove that in any triangle the following inequality holds:

$$\frac{1}{m_b m_c} + \frac{1}{m_c m_a} + \frac{1}{m_a m_b} \leqslant \frac{\sqrt{3}}{S}.$$

Gheorghe Szöllösy

11th GRADE

PROBLEM 1. Let A be an $2n$-order matrix with integer entries such that all entries of the principal diagonal have different parity from the rest of the entries.

Prove that $\det A$ cannot be zero.

Dinu Şerbănescu

PROBLEM 2. Consider a sequence $(a_n)_{n\geqslant 1}$ of positive real numbers such that

$$(a_{n+1} - a_n)^2 = a_n \quad \text{for all } n \geqslant 1.$$

Find the limit $L = \lim\limits_{n\to\infty} a_n$, knowing that this exists.

Valentin Vornicu

PROBLEM 3. (a) Find all 3-order real matrices which commutes with

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) Let n be a positive integer. Solve in $\mathcal{M}_3(\mathbf{R})$ the equation $X^n = A$.

Laurenţiu Panaitopol

PROBLEM 4. Consider the sequence $a_n = \sum\limits_{k=1}^{n} \frac{k}{2^k}$, $n \geqslant 1$. Prove that $\lim\limits_{n\to\infty} a_n = 2$ and $\lim\limits_{n\to\infty} \frac{2^n}{\sqrt[n]{n!}}(2 - a_n) = \mathrm{e}$.

Virgil Nicula

PROBLEM 5. Let a be a real number and let $f : \mathbf{R} \to \mathbf{R}$ be a function such that

$$f(x) \cdot f(y) + f(x) + f(y) = f(xy) + a \quad \text{for all } x, y \in \mathbf{R}.$$

(a) If f is bijective, find the number a and compute $f(-1)$, $f(0)$ and $f(1)$.

(b) Find all functions f which are continuous and bijective.

Marcel Chiriţă

12th GRADE

PROBLEM 1. Let $(G, \cdot)$ be a group and consider H a proper subset of G such that:

If $x \in H$ and $y \in G \setminus H$, then $xy \in G \setminus H$.

Prove that H is a subgroup of G.

Marcel Ţena

PROBLEM 2. Compute $\int \frac{e^x(x-2)}{x(x^2+e^x)}\,\mathrm{d}x$, $x \in (0, \infty)$.

Ioan V. Maftei

PROBLEM 3. Consider the function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = (x^2+1)\mathrm{e}^x$. Find the limit $\lim\limits_{n\to\infty} n \int_0^1 \left(f\left(\frac{x^2}{n}\right) - 1\right) \mathrm{d}x$.

* * *

PROBLEM 4. Suppose $A = \{f \in \mathbf{Q}[X] \mid f(n) \in \mathbf{Z} \text{ for all } n \in \mathbf{Z}\}$. Prove that:

(a) A is a subring of $\mathbf{Q}[X]$;

(b) $\mathbf{Z}[X]$ is not isomorphic to A.

* * *

I.2. SECOND ROUND – DISTRICT LEVEL

March 26, 2003

7th GRADE

PROBLEM 1. Find the disjoint sets B and C such that $B \cup C = \{1, 2, \dots, 10\}$ and the product of the elements of C equals the sum of elements of B.

Ioan Bogdan

PROBLEM 2. Consider a right triangle ABC ($m(\angle A) = 90°$). Let D be the intersection of the bisector line of A with the line BC, and P, Q the orthogonal projections of D onto lines AB and AC, respectively. If $BQ \cap DP = \{M\}$, $CP \cap DQ = \{N\}$ and $BQ \cap CP = \{H\}$, prove that:

(a) $PM = DN$;

(b) $MN \parallel BC$;

(c) $AH \perp BC$.

Mircea Fianu

PROBLEM 3. A grid consists of $2n$ vertical and $2n$ horizontal lines, each group disposed at equal distances. The lines are all painted in red and black, such that exactly n vertical and n horizontal lines are red.

Find the smallest n such that for any painting satisfying the above condition, there is a square formed by the intersection of two vertical and two horizontal lines, all of the same colour.

Radu Gologan

PROBLEM 4. Consider a triangle ABC. Let B' the reflection of B with respect to C, C' the reflection of C with respect to A and A' the reflection of A with respect to B.

(a) Prove that the area of $AC'A'$ is two times the area of ABC.

(b) If we erase the points A, B, C, is it possible to reconstruct them? Justify!

* * *

8th GRADE

PROBLEM 1. Let ABC be an equilateral triangle. The perpendiculars AA' and BB' on the plane containing ABC at the points A and B are $AA' = AB$ and $BB' = \frac{1}{2}AB$.

Find the angle between the planes (ABC) and $(A'B'C')$.

Neculai Solomon

PROBLEM 2. Let $M \subset \mathbf{R}$ be a finite set containing at least two elements. We say that the function f has property $\mathcal{P}$ if $f : M \to M$ and there are $a \in \mathbf{R}^*$ and $b \in \mathbf{R}$ such that $f(x) = ax + b$.

(a) Show that there is at least a function having property $\mathcal{P}$.

(b) Show that there are at most two functions having property $\mathcal{P}$.

(c) If M has 2003 elements with sum 0 and if there are two functions with property $\mathcal{P}$, prove that $0 \in M$.

Gabriel Popa

PROBLEM 3. Consider an array $n \times n$ ($n \geqslant 2$) with n^2 integers. In how many ways one can complete the array if the product of the numbers on any row and column is 5 or -5?

Mariana Coadă

PROBLEM 4. (a) Let MNP be a triangle with $\angle MNP > 60°$. Prove that MP is not the smallest side of the triangle.

(b) A plane contains an equilateral triangle ABC. The point V, that doesn't belong to the plane (ABC) is such that $\angle VAB = \angle VBC = \angle VCA$. Prove that if $VA = AB$, then all sides of the piramid $VABC$ are equal.

Valentin Vornicu

9th GRADE

PROBLEM 1. Find all functions $f : \mathbf{N}^* \to \mathbf{N}^*$ such that for any n, $n \geqslant 1$, the number

$$f(1) + f(2) + \cdots + f(n)$$

is the cube of a number at most equal to n.[1]

Lucian Dragomir

[1] $\mathbf{N}^*$ is the set of positive integers

PROBLEM 2. Find $n \in \mathbf{N}$, $n \geqslant 2$ and digits $a_1, a_2, \ldots, a_n$, such that

$$\sqrt{\overline{a_1a_2\ldots a_n}} - \sqrt{\overline{a_1a_2\ldots a_{n-1}}} = a_n.$$

($\overline{a_1a_2\ldots a_n}$ is the n-digit number with digits $a_1, a_2, \ldots, a_n$).

* * *

PROBLEM 3. On the blackboard there are given points A, B, C, D. Vlad constructs the points A', B', C', D' in the following manner: A' is the reflexion of A with respect to B, B' is the reflexion of B with respect to C, C' is the reflexion of C with respect to D and D' is the reflexion of D with respect to A. Maria eliminates from the blackboard the points A, B, C, D.

Can Vlad reconstruct the positions of these points? Justify; vectors can be used.

* * *

PROBLEM 4. A set A of nonzero vectors in the plane have property (S) if it consists of at least three elements and for any $\overrightarrow{u} \in A$ there are vectors $\overrightarrow{v}, \overrightarrow{w} \in A$ such that $\overrightarrow{v} \neq \overrightarrow{w}$ and $\overrightarrow{u} = \overrightarrow{v} + \overrightarrow{w}$.

(a) Prove that, for any $n \geqslant 6$, there is a set of vectors having property (S).

(b) Prove that any finite set of vectors with property (S) has at least six elements.

Mihai Băluna

10th GRADE

PROBLEM 1. In the interior of a cube there are 2003 points. Prove that, one can divide the cube in more than 2003^3 smaller cubes, such that any of the given points is in the interior of a small cube (not on the borders).

* * *

PROBLEM 2. Determine all functions $f : \mathbf{N}^* \to M$ having the property that

$$1 + f(n)f(n+1) = 2n^2\left(f(n+1) - f(n)\right),$$

for any $n \in \mathbf{N}^*$, in any of the situations

(a) $M = \mathbf{N}$;

(b) $M = \mathbf{Q}$.

Dinu Şerbănescu

PROBLEM 3. Let ABC be a triangle.

(a) Prove that if M is any point in its plane, then

$$AM \sin A \leqslant BM \sin B + CM \sin C.$$

(b) Let A_1, B_1, C_1 be points on the sides BC, AC and AB respectively, such that the angles of the triangle $A_1B_1C_1$ are in this order α, β, γ. Prove that

$$\sum AA_1 \sin\alpha \leqslant \sum BC \sin\alpha.$$

Dan Marinescu, Vasile Cornea

PROBLEM 4. Given positive numbers a, b, c, d such that $a > c > d > b > 1$ and $ab > cd$, prove that the function $f : [0, \infty) \to \mathbf{R}$, $f(x) = a^x + b^x - c^x - d^x$, is strictly increasing.

Cristinel Mortici

11th GRADE

PROBLEM 1. In the Cartesian plane xOy consider the collinear points $A_i(x_i, y_i)$, $i = \overline{1,4}$, such that there are invertible matrices $M \in \mathcal{M}_4(\mathbf{C})$ having the first two rows (x_1, x_2, x_3, x_4) and respectively (y_1, y_2, y_3, y_4).

Prove that for such a matrix M the sum of elements of M^{-1} is independent of M.

Marian Andronache

PROBLEM 2. Let $f : [0,1] \to [0,1]$ be a function that is continuous at the points 0 and 1, has limits from the left and from the right at any point and verifies $f(x-0) \leqslant f(x) \leqslant f(x+0)$ for any $x \in (0,1)$.

Prove that there is $x_0 \in (0,1)$ such that $f(x_0) = x_0$.

Mihai Piticari

PROBLEM 3. (a) Prove that any matrix $A \in \mathcal{M}_n(\mathbf{C})$ is the sum of n matrices of rank 1.

(b) Prove that I_n cannot be written as the sum of less than n matrices of rank 1.

Ion Savu, Manuela Prajea

PROBLEM 4. Let $\alpha > 1$ and let $f : [\frac{1}{\alpha}, \alpha] \to [\frac{1}{\alpha}, \alpha]$ a bijective function. Suppose that $f^{-1}(x) = \frac{1}{f(x)}$, for any $x \in [\frac{1}{\alpha}, \alpha]$. Prove that:

(a) f has at least a discontinuity point;

(b) if f is continuous at 1, than f has an infinity of discontinuity points;

(c) there is a function f verifying the given conditions and possesing only a finite number of discontinuity points.

Radu Miculescu

12^{th} GRADE

PROBLEM 1. Let $(G, \cdot)$ be a finite group with unity e. The least positive integer with the property that $x^n = e$, for any $x \in G$, is called the *exponent* of the group G.

(a) For any prime p, $p \geqslant 3$, show that the multiplicative group G_p consisting of those matrices of the form

$$\begin{pmatrix} \hat{1} & \hat{a} & \hat{b} \\ \hat{0} & \hat{1} & \hat{c} \\ \hat{0} & \hat{0} & \hat{1} \end{pmatrix} \quad \text{with } \hat{a}, \hat{b}, \hat{c} \in \mathbf{Z}_p,$$

is noncommutative and has exponent p.

(b) Show that, if $(G, \circ)$ and $(H, \bullet)$ are finite groups with exponents m respectively n, than the group $(G \times H, *)$ defined by $(g,h) * (g', h') = (g \circ g', h \bullet h')$, for any $(g,h), (g',h') \in G \times H$, has the exponent equal to the least common multiple of m and n.

(c) Deduce that any natural number n, $n \geqslant 3$ is the exponent of some finite noncommutative group.

* * *

PROBLEM 2. Consider two different continuous functions $f, g : [0,1] \to (0, \infty)$, such that $\int_0^1 f(x)\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x$.

Let $(x_n)_{n\geqslant 0}$ the sequence defined by $x_n = \int_0^1 \frac{(f(x))^{n+1}}{(g(x))^n}\,\mathrm{d}x$.

(a) Prove that $\lim\limits_{n\to\infty} x_n = \infty$.

(b) Show that the sequence $(x_n)_{n\geqslant 0}$ is increasing.

Dan Marinescu, Vasile Cornea

PROBLEM 3. Let K be a finite field in which the polynomial $X^2 - 5$ is irreducible in $K[X]$. Show that:

(a) $1 + 1 \neq 0$;

(b) for any $a \in K$, the polynomial $X^5 + a$ is reducible in $K[X]$.

Marian Andronache

PROBLEM 4. Consider the continuous functions $f : [0, \infty) \to \mathbf{R}$ and $g : [0,1] \to \mathbf{R}$. If $\lim\limits_{x\to\infty} f(x) = L \in \mathbf{R}$, prove that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n f(x) g\left(\frac{x}{n}\right) \mathrm{d}x = L \int_0^1 g(x)\,\mathrm{d}x.$$

Laurenţiu Panaitopol

I.3. FINAL ROUND

Sibiu, April 21, 2003

7^{th} GRADE

PROBLEM 1. Find the maximum number of elements which can be chosen from the set $\{1, 2, 3, \ldots, 2003\}$ such that the sum of any two chosen elements is not divisible by 3.

* * *

PROBLEM 2. Compute the maximum area of a triangle having a median of length 1 and a median of length 2.

* * *

PROBLEM 3. For every positive integer n consider

$$A_n = \sqrt{49n^2 + 0,35n}\,.$$

(a) Find the first three digits after the decimal point for A_1.

(b) Prove that the first three digits after the decimal point of A_n and A_1 are the same, for every n.

* * *

PROBLEM 4. In triangle ABC, P is the midpoint of side BC. Let $M \in (AB)$, $N \in (AC)$ be such that $MN \parallel BC$ and $\{Q\}$ be the common point of MP and BN. The perpendicular from Q on AC intersects AC in R and the parallel from B to AC in T. Prove that:

(a) $TP \parallel MR$;

(b) $\angle MRQ = \angle PRQ$.

Mircea Fianu

8th GRADE

PROBLEM 1. Let m, n be positive integers. Prove that the number $5^n + 5^m$ can be represented as sum of two perfect squares if and only if $n - m$ is even.

Vasile Zidaru

PROBLEM 2. In a meeting there are 6 participants. It is known that among them there are seven pairs of friends and in any group of three persons there are at least two friends. Prove that:

(a) there exists a person who has at least three friends;

(b) there exists three persons who are friends with each other.

Valentin Vornicu

PROBLEM 3. The real numbers a, b fulfil the conditions

(i) $0 < a < a + \frac{1}{2} \leqslant b$;

(ii) $a^{40} + b^{40} = 1$.

Prove that b has the first 12 digits after the decimal point equal to 9.

Mircea Fianu

PROBLEM 4. In tetrahedron $ABCD$, G_1, G_2 and G_3 are barycenters of the faces ACD, ABD and BCD respectively.

(a) Prove that the straight lines BG_1, CG_2 and AG_3 are concurent.

(b) Knowing that $AG_3 = 8$, $BG_1 = 12$ and $CG_2 = 20$ compute the maximum possible value of the volume of $ABCD$.

* * *

9th GRADE

PROBLEM 1. Find positive integers a, b if for every $x, y \in [a, b]$, $\frac{1}{x} + \frac{1}{y} \in [a, b]$.

* * *

PROBLEM 2. An integer n, $n \geqslant 2$ is called ***friendly*** if there exists a family $A_1, A_2, \ldots, A_n$ of subsets of the set $\{1, 2, \ldots, n\}$ such that:

(1) $i \notin A_i$ for every $i = \overline{1, n}$;

(2) $i \in A_j$ if and only if $j \notin A_i$, for every distinct $i, j \in \{1, 2, \ldots, n\}$;

(3) $A_i \cap A_j$ is non-empty, for every $i, j \in \{1, 2, \ldots, n\}$.

Prove that:

(a) 7 is a friendly number;

(b) n is friendly if and only if $n \geqslant 7$.

Valentin Vornicu

PROBLEM 3. Prove that the midpoints of the altitudes of a triangle are collinear if and only if the triangle is right.

Dorin Popovici

PROBLEM 4. Let P be a plane. Prove that there exists no function $f : P \to P$ such that for every convex quadrilateral $ABCD$, the points $f(A), f(B), f(C), f(D)$ are the vertices of a concave quadrilateral.

Dinu Şerbănescu

10th GRADE

PROBLEM 1. Let $OABC$ be a tetrahedron such that $OA \perp OB \perp OC \perp OA$, r be the radius of its inscribed sphere and H be the orthocenter of triangle ABC. Prove that $OH \leqslant r(\sqrt{3} + 1)$.

* * *

PROBLEM 2. The complex numbers z_i, $i = 1, \ldots, 5$, have the same non-nil modulus and $\sum_{i=1}^{5} z_i = \sum_{i=1}^{5} z_i^2 = 0$.

Prove that $z_1, z_2, \ldots, z_5$ are the complex coordinates of the vertices of a regular pentagon.

* * *

PROBLEM 3. Let a, b, c be the complex coordinates of the vertices A, B, C of a triangle. It is known that $|a| = |b| = |c| = 1$ and that there exists $\alpha \in (0, \frac{\pi}{2})$ such that $a + b\cos\alpha + c\sin\alpha = 0$. Prove that $1 < \text{area}(ABC) \leqslant \frac{1+\sqrt{2}}{2}$.

Gheorghe Iurea

PROBLEM 4. A finite set A of complex numbers has the property: $z \in A$ implies $z^n \in A$ for every positive integer n.

(a) Prove that $\sum_{z \in A} z$ is an integer.

(b) Prove that for every integer k one can chose a set A which fulfils the above condition and $\sum_{z \in A} z = k$.

Paltin Ionescu

11^{th} GRADE

PROBLEM 1. Find the locus of the points M from the plane of a rhombus $ABCD$ such that

$$MA \cdot MC + MB \cdot MD = AB^2.$$

Ovidiu Pop

PROBLEM 2. Consider the real numbers $1 \leqslant a_1 < a_2 < a_3 < a_4$, $x_1 < x_2 < x_3 < x_4$ and the matrix $M = (a_i^{x_j})_{i,j\in\overline{1,4}}$.

Prove that $\det M > 0$.

* * *

PROBLEM 3. The real functions f, g are such that f is continuous and g is increasing and unbounded. It is known that for every sequence (x_n) of rational numbers with $(x_n)_n \to \infty$, $\lim\limits_{n\to\infty} f(x_n)g(x_n) = 1$.

Prove that $\lim\limits_{x\to\infty} f(x)g(x) = 1$.

Radu Gologan

PROBLEM 4. Let A be a 3×3 matrix with real entries. Prove that:

(a) if f is a real polynomial without real roots then $f(A) \neq 0_3$;

(b) there exists a positive integer n such that

$$(A + A^*)^{2n} = A^{2n} + (A^*)^{2n}$$

if and only if $\det A = 0$.

Laurenţiu Panaitopol

12^{th} GRADE

PROBLEM 1. (a) Let K be a field and $n \geqslant 2$ be an integer. Describe the set

$$Z(\mathcal{M}_n(K)) = \{A \in \mathcal{M}_n(K) \mid AX = XA \text{ for every } X \in \mathcal{M}_n(K)\}$$

and prove that the ring $Z(\mathcal{M}_n(K))$ is isomorphic to K.

(b) Prove that the rings $\mathcal{M}_n(\mathbf{R})$ and $\mathcal{M}_n(\mathbf{C})$ are not isomorphic.

* * *

PROBLEM 2. Let $n \geqslant 3$ be an odd integer. Find all continuous functions $f : [0,1] \to \mathbf{R}$ such that

$$\int_0^1 \left(f\left(\sqrt[k]{x}\right)\right)^{n-k} \mathrm{d}x = \frac{k}{n},$$

for every $k \in \{1, \ldots, n-1\}$.

Titu Andreescu

PROBLEM 3. A continuous function $f : \mathbf{R} \to \mathbf{R}$ fulfils the condition $xf(x) \geqslant \int_0^x f(t)\,\mathrm{d}t$, for every real x.

(a) Prove that the function $g : \mathbf{R}^* \to \mathbf{R}$, $g(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t$ is increasing on $(-\infty, 0)$ and on $(0, \infty)$.

(b) Prove that if f has also the property

$$\int_x^{x+1} f(t)\,\mathrm{d}t = \int_{x-1}^{x} f(t)\,\mathrm{d}t \quad \text{for all real } x,$$

then f is constant.

Mihai Piticari

PROBLEM 4. For a finite commutative group $(G, +)$ denote by $n(G)$ its cardinal and by $i(G)$ the number of algebraic operations $(G, *)$ such that $(G, +, *)$ is a ring (with unity). Prove that:

(a) $i(\mathbf{Z}_{12}) = 4$;

(b) $i(A \times B) \geqslant i(A)i(B)$, for every finite commutative groups A and B;

(c) there exist two sequences of finite commutative groups $(G_k)_{k\geqslant 1}$, $(H_k)_{k\geqslant 1}$ such that

$$\lim_{k\to\infty} \frac{n(G_k)}{i(G_k)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{n(H_k)}{i(H_k)} = \infty.$$

Barbu Berceanu

Part II. THE 54th NATIONAL MATHEMATICAL OLYMPIAD SOLUTIONS

II.1. FIRST ROUND – CITY OF BUCHAREST

9th GRADE

PROBLEM 1. Find the integer part of the number

$$\left.\sqrt[3]{24+\sqrt[3]{24+\cdots+\sqrt[3]{24}}}\ \right\}\ n \text{ roots},$$

where $n \geqslant 1$.

Solution. Set $a_n = \left.\sqrt[3]{24+\sqrt[3]{24+\cdots+\sqrt[3]{24}}}\ \right\}\ n$ roots for any $n \geqslant 1$. We have $2 \leqslant \sqrt[3]{24} = a_1 < 3$ and $a_{n+1} = \sqrt[3]{24+a_n}$. By induction on n it is easy to prove that

$$2 \leqslant a_n < 3 \quad \text{for all } n \geqslant 1.$$

Indeed, $a_{n+1} = \sqrt[3]{24+a_n} < \sqrt[3]{24+3} = 3$ and $a_n > \sqrt[3]{24} > 2$. Thus $[a_n]$=2 for all $n \geqslant 1$.

PROBLEM 2. Let x and y be real numbers so that x^2+y^2, x^3+y^3 and x^4+y^4 are rational numbers. Prove that $x+y$ and xy are also rationals.

Solution. For $x=0$ or $y=0$ the claim is obvious. For $xy \neq 0$, notice that

$$(1) \qquad x^2y^2 = \frac{1}{2}\left[(x^2+y^2)^2 - (x^4+y^4)\right] \in \mathbf{Q}.$$

On the other hand, $x^6+y^6 = (x^2+y^2)^3 - 3x^2y^2(x^2+y^2) \in \mathbf{Q}$ and then

$$(2) \qquad x^3y^3 = \frac{1}{2}\left[(x^3+y^3)^2 - (x^6+y^6)\right] \in \mathbf{Q}.$$

From (1) and (2) we derive that

$$xy = \frac{x^3y^3}{x^2y^2} \in Q.$$

Finally, $x+y = \frac{x^3+y^3}{(x^2+y^2)-xy} \in \mathbf{Q}$.

PROBLEM 3. Find the function $f : \mathbf{R} \to \mathbf{R}$ with the properties:
(a) $f(x) = x^2$ for all $x \in [0,1)$;
(b) $f(x+1) = f(x) + x$ for all $x \in \mathbf{R}$.

Solution. Let n be a positive integer. Then

$$\begin{aligned} f(x+n) &= f(x+n-1) + (x+n-1) \\ &= f(x+n-2) + (x+n-2) + (x+n-1) \\ &\ \ \vdots \\ &= f(x) + x + (x+1) + \cdots + (x+n-1) \\ &= f(x) + \frac{n(2x+n-1)}{2} \quad \text{for all } x \in \mathbf{R}. \end{aligned}$$

Consider $x < 0$ an arbitrary real number and $n = -[x]$, $n \in \mathbf{N}^*$. Since $x+n = x-[x] = \{x\} \in [0,1)$, it follows that $f(x+n) = (x+n)^2 = \{x\}^2$ and

$$\begin{aligned} f(x) &= f(x+n) - \frac{n(2x+n-1)}{2} = \{x\}^2 - \frac{[x](2x-[x]-1)}{2} \\ &= \frac{2\{x\}^2 + (x-\{x\})(x+\{x\}-1)}{2} = \frac{x^2+\{x\}^2-x+\{x\}}{2} \end{aligned}$$

for all real $x < 0$.

Now, consider a real number $x \geqslant 1$ and let $n = [x] \in \mathbf{N}^*$. We have

$$\begin{aligned} f(x) &= f(\{x\}+n) = f(\{x\}) + \frac{n(2\{x\}+n-1)}{2} \\ &= \{x\}^2 + \frac{[x](x+\{x\}-1)}{2} = \frac{x^2+\{x\}^2-x+\{x\}}{2} \end{aligned}$$

for all real $x \geqslant 1$.

Finally, as $x^2 = \frac{x^2+\{x\}^2-x+\{x\}}{2}$ for any $x \in [0,1)$, then

$$f(x) = \frac{x^2+\{x\}^2-x+\{x\}}{2} \quad \text{for all } x \in \mathbf{R}.$$

PROBLEM 4. Consider the points A, B, C, D in a plane, not three of them collinear. Points H_1 and H_2 are the orthocenters of the triangles ABC and ABD, respectively.

Prove that A, B, C, D lie on the same circle if and only if

$$\overrightarrow{H_1H_2} = \overrightarrow{CD}.$$

Solution. Let O_1 and O_2 be the circumcenters of the triangles ABC and ABD. By Sylvester formula,

$$\overrightarrow{O_1H_1} = \overrightarrow{O_1A} + \overrightarrow{O_1B} + \overrightarrow{O_1C} \quad \text{and} \quad \overrightarrow{O_2H_2} = \overrightarrow{O_2A} + \overrightarrow{O_2B} + \overrightarrow{O_2D}.$$

Subtracting the relations implies

$$\begin{aligned} &\overrightarrow{O_1H_1} - \overrightarrow{O_2H_2} = 2\overrightarrow{O_1O_2} + \overrightarrow{O_1C} - \overrightarrow{O_2D} \\ &\Leftrightarrow \overrightarrow{O_1H_1} - (\overrightarrow{O_2O_1} - \overrightarrow{O_1H_2}) = 2\overrightarrow{O_1O_2} + \overrightarrow{O_1C} - (\overrightarrow{O_2O_1} + \overrightarrow{O_1D}) \\ &\Leftrightarrow \overrightarrow{H_2H_1} = 2\overrightarrow{O_1O_2} + \overrightarrow{DC} \\ &\Leftrightarrow \overrightarrow{H_1H_2} - \overrightarrow{CD} = 2\overrightarrow{O_2O_1}. \end{aligned}$$

Then $O_1 = O_2 \Leftrightarrow \overrightarrow{H_1H_2} = \overrightarrow{CD}$, as desired.

PROBLEM 5. An arbitrary point M is considered on the side BC of the triangle ABC. Let $\mathcal{C}_0, \mathcal{C}_2, \mathcal{C}_3$ be the incircles of the triangles ABC, ABM, ACM, respectively.

(a) Prove that $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent if and only if $M \in \mathcal{C}_0$.

(b) Suppose $M \in \mathcal{C}_0$ and let S and D be the midpoints of the segments AM and BC. Prove that $p\overrightarrow{AI} = a\overrightarrow{AS} + (p-a)\overrightarrow{AD}$. Furthermore, derive that points I, S, D are collinear and $\frac{IS}{ID} = \frac{p-a}{a}$.

Solution. (a) First recall a usefull result:

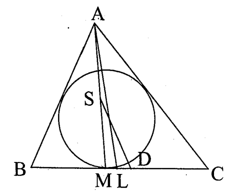
Suppose the incircle of triangle ABC touch the sides AB, BC, CA at points C_1, A_1, B_1, respectively. Then

$$AB_1 = AC_1 = \frac{AB + AC - BC}{2} \quad (= p - a).$$

Let the line AM touches the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ at T_1 and T_2, respectively. The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent if and only if $T_1 = T_2$; that is when $AT_1 = AT_2$. Using the above result, this is equivalent to

$$\begin{aligned} \frac{AM + AB - BM}{2} = \frac{AM + AC - CM}{2} &\Leftrightarrow AB - BM = AC - (BC - BM) \\ &\Leftrightarrow BM = \frac{BM + AC - AC}{2} \\ &\Leftrightarrow M \in \mathcal{C}_0, \end{aligned}$$

as needed.



(b) Let a, b, c be the side lengths of the triangle and let L be the intersection point of BC with the internal bisector AI of $\angle BAC$.

By angle bisector theorem, $\frac{LB}{LC} = \frac{AB}{AC}$ and $\frac{LB}{BC} = \frac{AB}{AB+AC}$.

The key idea is to represent vectors $\overrightarrow{AD}, \overrightarrow{AS}$ and $\overrightarrow{AI}$ as linear combinations of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Thus,

$$\overrightarrow{AD} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}),$$

$$\overrightarrow{AS} = \frac{1}{2}(\overrightarrow{AM}) = \frac{1}{2}\Big(\frac{CM}{BC}\cdot\overrightarrow{AB} + \frac{BM}{BC}\cdot\overrightarrow{AC}\Big) = \frac{(p-c)\overrightarrow{AB} + (p-b)\overrightarrow{AC}}{2a}$$

and

$$\overrightarrow{AL} = \frac{LC}{BC}\cdot\overrightarrow{AB} + \frac{LB}{BC}\cdot\overrightarrow{AC} = \frac{AC}{AB+AC}\cdot\overrightarrow{AB} + \frac{AB}{AB+AC}\cdot\overrightarrow{AC} = \frac{b\overrightarrow{AB} + c\overrightarrow{AC}}{b+c}.$$

As BI is the bisector of $\angle ABL$, then

$$\frac{AI}{IL} = \frac{AB}{BL} = \frac{c}{\frac{ac}{b+c}} = \frac{b+c}{a} \quad \text{and} \quad \frac{AI}{AL} = \frac{b+c}{2p}.$$

Consequently,

$$\overrightarrow{AI} = \frac{b\overrightarrow{AB} + c\overrightarrow{AC}}{2p}.$$

A short algebraic manipulation shows that $p\overrightarrow{AI} = a\overrightarrow{AS} + (p-a)\overrightarrow{AB}$. Since $\frac{a}{p} + \frac{p-a}{p} = 1$, it follows that point I lies on the line segment SD such that $\frac{IS}{ID} = \frac{p-a}{a}$.

10^{th} GRADE

PROBLEM 1. Let $ABCD$ be a convex quadrilateral and let M be the midpoint of the side CD. Lines BM and AM are perpendicular and $AB = BC + AD$. Prove that lines BC and AD are parallel.

Solution. Reflect M at P across the midpoint of the segment AB. The angle $\angle AMB$ of the parallelogram $AMBP$ is right, hence $AMBP$ is a rectangle and

$$(1) \qquad MP = AB = AD + BC.$$

On the other hand

$$(2) \qquad \overrightarrow{MP} = \overrightarrow{MA} + \overrightarrow{MB} = (\overrightarrow{MD} + \overrightarrow{DA}) + (\overrightarrow{MC} + \overrightarrow{CB}) = \overrightarrow{DA} + \overrightarrow{CB},$$

since $\overrightarrow{MD} + \overrightarrow{MC} = \overrightarrow{0}$.

The relations (1) and (2) give $AD \parallel BC \parallel MP$, as needed.

PROBLEM 2. Let a, b, c, d be complex numbers with equal absolute values such that $a + b + c = d$.

Prove that d is equal to one of the numbers a, b or c.

Solution. It suffices to prove that $(d-a)(d-b)(d-c) = 0$, which is equivalent to

$$(1) \quad d^3 - d^2(a+b+c) + d(ab+bc+ca) - abc = 0 \quad \text{or} \quad d(ab+bc+ca) = abc.$$

Put $r = |a| = |b| = |c| = |d|$. Then $r^2 = a\overline{a} = b\overline{b} = c\overline{c} = d\overline{d}$. The condition $d = a+b+c$ gives $\overline{d} = \overline{a} + \overline{b} + \overline{c}$, then $\frac{r^2}{d} = \frac{r^2}{a} + \frac{r^2}{b} + \frac{r^2}{c}$. By multiplying out, the last equality reduces to (1), as desired.

PROBLEM 3. Let n and p be positive integers with $p > 2^n$. Prove that the integer part of the number $\sum_{k=0}^{n} \sqrt[p]{1 + \binom{n}{k}}$ is equal to $n+1$.

Solution. We have to prove that

$$n + 1 \leqslant \sum_{k=0}^{n} \sqrt[p]{1 + \binom{n}{k}} < 2.$$

The left inequality is obvious, since $\sqrt[p]{1 + \binom{n}{k}} > 1$. For the second we use Bernoulli inequality. We have

$$\left(1 + \frac{1}{p}\binom{n}{k}\right)^p \geqslant 1 + p \cdot \frac{1}{p}\binom{n}{k} = 1 + \binom{n}{k},$$

hence $1 + \frac{1}{p}\binom{n}{k} \geqslant \sqrt[p]{1 + \binom{n}{k}}$ for all $k = \overline{0, n}$. Summing up yiels

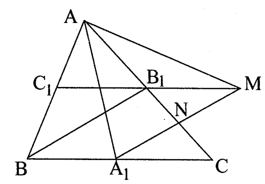
$$\sum_{k=0}^{n} \sqrt[p]{1 + \binom{n}{k}} \leqslant n + 1 + \frac{1}{p}\sum_{k=0}^{n} \binom{n}{k} = n + 1 + \frac{2^n}{p} < n + 2,$$

as needed.

PROBLEM 4. Prove that in any triangle the following inequality holds:

$$\frac{1}{m_b m_c} + \frac{1}{m_c m_a} + \frac{1}{m_a m_b} \leqslant \frac{\sqrt{3}}{S}.$$

Solution. With the medians of a triangle one can form another triangle, as shown below:



($A_1M = BB_1$ and $AM = CC_1$).

Furthermore, the area of the triangle formed by medians is $S_m = \frac{3}{4}S$. Indeed,

$$\frac{S_m}{S} = \frac{2\,\text{area}[AA_1N]}{2\,\text{area}[AA_1C]} = \frac{AN}{AC} = \frac{3}{4}.$$

Thus, the inequality becomes

$$(1) \qquad \frac{1}{m_a m_b} + \frac{1}{m_b m_c} + \frac{1}{m_c m_a} \leqslant \frac{3\sqrt{3}}{4S_m}.$$

Writting a, b, c, S for m_a, m_b, m_c, S_m, the inequality (1) reduces to

$$\frac{S}{ab} + \frac{S}{bc} + \frac{S}{ca} \leqslant \frac{3\sqrt{3}}{4},$$

or

$$\sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{4},$$

which is a known inequality. (For a short proof, notice that the sine function is concave down on $[0, \pi]$.)

11th GRADE

PROBLEM 1. Let A be an $2n$-order matrix with integer entries such that all entries of the principal diagonal have different parity from the rest of the entries. Prove that $\det A$ cannot be zero.

Solution. First, assume that all entries of the principal diagonal are odd, and the rest are even. Then $A \equiv I_{2n} (\text{mod}\, 2)$, hence $\det A \equiv 1 (\text{mod}\, 2)$. Thus, $\det A$ is an odd number, so $\det A \neq 0$.

Now, assume that the numbers of the principal diagonal are even, and the rest are odd. Notice that A^2 has all the entries of the principal diagonal odd and the rest even, so again $\det A^2$ is an odd number. Since $(\det A)^2 = \det A^2$, the claim follows.

PROBLEM 2. Consider a sequence $(a_n)_{n \geqslant 1}$ of positive real numbers such that

$$(a_{n+1} - a_n)^2 = a_n \quad \text{for all } n \geqslant 1.$$

Find the limit $L = \lim\limits_{n \to \infty} a_n$, knowing that this exists.

Solution. Suppose $L \in \mathbf{R}$. Then $(L - L)^2 = L$, hence $L = 0$. We prove that this assumption leads to contradiction.

Put $b_n = \frac{a_{n+1}}{a_n}$, $n \geqslant 1$. The relation $(a_{n+1} - a_n)^2 = a_n$ becomes $(b_n - 1)^2 = \frac{1}{a_n}$. As $b_n \geqslant |b_n - 1| - 1 = \frac{1}{\sqrt{a_n}} - 1 \to \infty$, by d'Alembert criterion yields $a_n \to \infty$, contradiction.

Thus $L \notin \mathbf{R}$, and since $a_n > 0$ we obtain $L = \infty$.

PROBLEM 3. (a) Find all 3-order real matrices which commutes with

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) Let n be a positive integer. Solve in $\mathcal{M}_3(\mathbf{R})$ the equation $X^n = A$.

Solution. (a) Consider $X = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. From $AX = XA$ follows that $d = g = h = 0$, $a = e = i$, $f = 3b$, thus

$$X = \begin{pmatrix} a & b & c \\ 0 & a & 3b \\ 0 & 0 & a \end{pmatrix}, \quad a, b, c \in \mathbf{R}.$$

(b) Let X be a solution of the equation $X^n = A$. Then $AX = X^n \cdot X = X \cdot X^n = XA$, hence

$$X = \begin{pmatrix} a & b & c \\ 0 & a & 3b \\ 0 & 0 & a \end{pmatrix}.$$

Since $\det(X^n) = \det A = 0$, we have $\det X = 0$. Hence $a = 0$ and $X = \begin{pmatrix} 0 & b & c \\ 0 & 0 & 3b \\ 0 & 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 0 & 3b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $X^3 = 0_3$. As $X^2 \neq A$ and $X^n = 0^3 \neq A$ for $n \geqslant 3$, we infer that the equation has no solution for $n \geqslant 2$. For $n = 1$, obviously $X = A$.

PROBLEM 4. Consider the sequence $a_n = \sum\limits_{k=1}^{n} \frac{k}{2^k}$, $n \geqslant 1$. Prove that $\lim\limits_{n \to \infty} a_n = 2$ and $\lim\limits_{n \to \infty} \frac{2^n}{\sqrt[n]{n!}}(2 - a_n) = \mathrm{e}$.

Solution. We have

$$\begin{aligned}\frac{1}{2}a_n = a_n - \frac{1}{2}a_n &= \sum_{k=1}^{n} \frac{k}{2^k} - \sum_{k=1}^{n} \frac{k}{2^{k+1}} = \sum_{k=1}^{n} \frac{k}{2^k} - \sum_{k=1}^{n} \frac{k+1}{2^{k+1}} + \sum_{k=1}^{n} \frac{1}{2^{k+1}} \\ &= \sum_{k=1}^{n} \frac{1}{2^k} - \frac{n+1}{2^{n+1}} = 1 - \frac{n+2}{2^{n+1}},\end{aligned}$$

hence $a_n = 2 - \frac{n+2}{2^n}$ and $\lim\limits_{n \to \infty} a_n = 2$.

Furthermore, $2^{n+1} - 2^n a_n = n + 2$ and

$$\lim_{n \to \infty} \frac{2^n (2 - a_n)}{\sqrt[n]{n!}} = \lim_{n \to \infty} \frac{n+2}{\sqrt[n]{n!}} = \lim_{n \to \infty} \left(\frac{n}{\sqrt[n]{n!}} + \frac{2}{\sqrt[n]{n!}} \right) = \mathrm{e}.$$

PROBLEM 5. Let a be a real number and let $f : \mathbf{R} \to \mathbf{R}$ be a function such that

$$f(x) \cdot f(y) + f(x) + f(y) = f(xy) + a \quad \text{for all } x, y \in \mathbf{R}.$$

(a) If f is bijective, find the number a and compute $f(-1)$, $f(0)$ and $f(1)$.

(b) Find all functions f which are continuous and bijective.

Solution. Setting $y = 1$ yields $f(x) \cdot f(1) = a - f(1)$. If $f(1) \neq 0$, then f is a constant function, false. Thus $f(1) = 0$ and then $a = 0$.

Setting $x = y = -1$ yields $f^2(-1) + 2f(-1) = 0$. Since $f(-1) \neq f(1) = 0$, we obtain $f(-1) = -2$. Finally, put $y = 0$ and $x = -1$. It follows that $f(0) = -1$.

(b) Consider the function $g : \mathbf{R} \to \mathbf{R}$, $g(x) = f(x) + 1$. As $(f(x)+1)\cdot(f(y)+1) = f(xy)+1$, we have

$$g(x) \cdot g(y) = g(xy).$$

Moreover, $g(0) = 0$ and $g(-1) = -1$, hence $g(x^2) = g^2(x) \geqslant 0$ and $g(-x) = -g(x)$; in other words, g is an odd function and $g(x) > 0$ for all real $x > 0$.

Let $h(x) = \ln g(\mathrm{e}^x)$ for $x \in \mathbf{R}$. Then

$$h(x) + h(y) = \ln g(\mathrm{e}^x) + \ln g(\mathrm{e}^y) = \ln(g(\mathrm{e}^x)g(\mathrm{e}^y)) = \ln g(\mathrm{e}^{x+y}) = h(x+y)$$

for all $x, y \in R$. Since h is continuous, we have $h(x) = bx$ for some $b \in \mathbf{R}^*$. Then $g(\mathrm{e}^x) = \mathrm{e}^{bx}$ and so $g(x) = x^b$ for $x > 0$. Using the continuity of g yields $0 = g(0) = \lim_{x \searrow 0} x^b$, thus $b > 0$. Consequently,

$$g(x) = \begin{cases} x^b, & x \geqslant 0, \\ -(-x)^b, & x < 0, \end{cases}$$

then

$$f(x) = \begin{cases} x^b - 1, & x \geqslant 0, \\ -(-x)^b - 1, & x < 0 \end{cases}$$

for some $b > 0$.

12th GRADE

PROBLEM 1. Let $(G, \cdot)$ be a group and consider H a proper subset of G such that:

If $x \in H$ and $y \in G \setminus H$, then $xy \in G \setminus H$.

Prove that H is a subgroup of G.

Solution. To prove that H is a subgroup of G it suffices to show that $x^{-1}y \in H$, for all $x, y \in H$.

Let $x, y \in H$ and assume by contradiction that $x^{-1}y \in G \setminus H$. Then $y = x(x^{-1}y) \in G \setminus H$, a contradiction.

PROBLEM 2. Compute $\int \frac{\mathrm{e}^x(x-2)}{x(x^2+\mathrm{e}^x)}\,\mathrm{d}x$, $x \in (0, \infty)$.

Solution. We have

$$\begin{aligned}\int \frac{\mathrm{e}^x(x-2)}{x(x^2+\mathrm{e}^x)}\,\mathrm{d}x &= \int \frac{3x^2 + \mathrm{e}^x + x\mathrm{e}^x - 3x^2 - 3\mathrm{e}^x}{x(x^2+\mathrm{e}^x)}\,\mathrm{d}x \\ &= \int \frac{(x^3 + x\mathrm{e}^x)'}{x^3 + x\mathrm{e}^x}\,\mathrm{d}x - 3\int \frac{1}{x}\,\mathrm{d}x \\ &= \ln(x^3 + x\mathrm{e}^x) - 3\ln x + C = \ln\frac{x^2+\mathrm{e}^x}{x^2} + C.\end{aligned}$$

PROBLEM 3. Consider the function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = (x^2+1)\mathrm{e}^x$. Find the limit $\lim_{n\to\infty} n\int_0^1 \left(f\left(\frac{x^2}{n}\right) - 1\right)\mathrm{d}x$.

Solution. We have

$$\begin{aligned} n\int_0^1 \left(f\left(\frac{x^2}{n}\right) - 1\right)\mathrm{d}x &= n\int_0^1 \left[\left(\frac{x^4}{n^2} + 1\right)\mathrm{e}^{\frac{x^2}{n}} - 1\right]\mathrm{d}x \\ &= \frac{1}{n}\int_0^1 x^4\mathrm{e}^{\frac{x^2}{n}}\,\mathrm{d}x + n\int_0^1 \left(\mathrm{e}^{\frac{x^2}{n}} - 1\right)\mathrm{d}x.\end{aligned}$$

As $0 \leqslant \frac{1}{n}\int_0^1 x^4\mathrm{e}^{\frac{x^2}{n}}\,\mathrm{d}x \leqslant \frac{\mathrm{e}}{n}\int_0^1 x^4\,\mathrm{d}x = \frac{\mathrm{e}}{5n}$, it follows that $\lim_{n\to\infty}\frac{1}{n}\int_0^1 x^4\mathrm{e}^{\frac{x^2}{n}}\,\mathrm{d}x = 0$.

Hence it remains to compute $\lim_{n\to\infty} n\int_0^1\left(\mathrm{e}^{\frac{x^2}{n}} - 1\right)\mathrm{d}x$. Since $\frac{x^2}{n} \leqslant \mathrm{e}^{\frac{x^2}{n}} - 1 < \frac{x^2}{n} + \frac{\mathrm{e}}{2}\cdot\frac{x^4}{n^2}$ for all $x \in [0,1]$, then $\frac{1}{3} \leqslant n\int_0^1\left(\mathrm{e}^{\frac{x^2}{n}} - 1\right)\mathrm{d}x < \frac{1}{3} + \frac{\mathrm{e}}{10n}$. By the squeeze theorem we obtain $\lim_{n\to\infty} n\int_0^1\left(\mathrm{e}^{\frac{x^2}{n}} - 1\right)\mathrm{d}x = \frac{1}{3}$.

PROBLEM 4. Suppose $A = \{f \in \mathbf{Q}[X] \mid f(n) \in \mathbf{Z} \text{ for all } n \in \mathbf{Z}\}$. Prove that:

(a) A is a subring of $\mathbf{Q}[X]$;

(b) $\mathbf{Z}[X]$ is not isomorphic to A.

Solution. (a) Let $f, g \in A$. Since $(f-g)(n) = f(n) - g(n) \in \mathbf{Z}$, $(fg)(n) = f(n)g(n) \in \mathbf{Z}$ for all integers n, and $1 \in A$, the conclusion follows.

(b) Suppose by contradiction that $F : \mathbf{Z}[X] \to A$ is an isomorphism. From $F(1) = 1$ and $F(f+g) = F(f) + F(g)$ for $f, g \in \mathbf{Z}[X]$ we derive that $F(a) = a$ for all $a \in \mathbf{Z}$. Let $h \in A$ such that $F(X) = h$. Then $F(a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0) = a_nh^n + a_{n-1}h^{n-1} + \cdots + a_1h + a_0$ for $n \geqslant 0$ and $a_0, a_1, \ldots, a_n \in \mathbf{Z}$. Consequently, $\deg F(f) = \deg f \cdot \deg h$, for all $f \in \mathbf{Z}[X]$. The function F is surjective, hence there exists $f_0 \in \mathbf{Z}[X]$ with $F(f_0) = X$. Then $1 = \deg x = \deg F(f_0) = \deg f_0 \cdot \deg h$, so $\deg h = 1$ and $h = aX + b$. Moreover, $b = h(0) \in \mathbf{Z}$ and $a + b = h(1) \in \mathbf{Z}$, therefore $h \in \mathbf{Z}[X]$ and $F(f) \in \mathbf{Z}[X]$ for all $f \in \mathbf{Z}[X]$. Since f is surjective, $A = \mathbf{Z}[X]$. This is contradiction, since $\frac{X(X+1)}{2} \in A \setminus \mathbf{Z}[X]$.

Alternative solution. It is easy to prove that the polynomials $\frac{X(X+1)}{2}$, $X+1$ and 2 are irreducible in A. It follows that $X^2 + X$ has two decompositions in irreducible factors: $X \cdot (X+1)$ and $2 \cdot \frac{X(X+1)}{2}$. Thus A is not a factorial ring, while $\mathbf{Z}[X]$ is factorial; contradiction.

II.2. SECOND ROUND – DISTRICT LEVEL

7th GRADE

PROBLEM 1. Find the disjoint sets B and C such that $B \cup C = \{1, 2, \ldots, 10\}$ and the product of the elements of C equals the sum of elements of B.

Solution. Remark that the sum of the elements in B is not greater than $1 + 2 + \cdots + 10 = 55$, that is C has at most four elements. Otherwise the product of elements in C should be at least $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

We are imposed to consider the following situations:

(i) C consists of a single element. This is obviously not possible as the product of elements in C is at most 9 and the sum of elements in B is at least $55 - 9 - 46$.

(ii) C consists of two elements x, y. Suppose $x < y$. One obtains $xy = 55 - x - y$ which can be written $(x+1)(y+1) = 56$. As $x + 1 < y + 1 \leqslant 11$, the only possibility is $x + 1 = 7$ and $y + 1 = 8$, giving the sets $C = \{6, 7\}$ and $B = \{1, 2, 3, 4, 5, 8, 9, 10\}$.

(iii) C consists of three elements, say $x < y < z$. The given condition becomes $xyz = 55 - x - y - z$.

For $x = 1$, using the same techniques as above we obtain $y = 4$ and $z = 10$, thus $C = \{1, 4, 10\}$ and $B = \{2, 3, 5, 6, 7, 8, 9\}$.

For $x = 2$, we have $2yz + y + z = 53$, or $(2y+1)(2z+1) = 107$, a prime number. Thus there are no solutions in this case.

For $x \geqslant 3$ it is easy to see that $xyz \geqslant 3 \cdot 4 \cdot 5 = 60 > 55 - x - y - z$. No solutions.

(iv) C consists of four elements $x < y < y < t$. We are forced for $x = 1$; otherwise $xyzt \geqslant 2 \cdot 3 \cdot 4 \cdot 5 = 120 > 55$. We have $yzt = 54 - y - z - t$ and $2 \leqslant y < z < t$. The case $y \geqslant 3$ implies a contradiction as in (iii). So $y = 2$ and $2zt + z + t = 52$ or $(2z+1)(2t+1) = 105$. Thus $2z + 1 = 7$ and $2t + 1 = 15$ which implies $z = 3$ and $t = 7$, giving the solutions $C = \{1, 2, 3, 7\}$ and $B = \{4, 5, 6, 8, 9, 10\}$.

PROBLEM 2. Consider a right triangle ABC ($m(\angle A) = 90°$). Let D be the intersection of the bisector line of A with the line BC, and P, Q the orthogonal

projections of D onto lines AB and AC, respectively. If $BQ \cap DP = \{M\}$, $CP \cap DQ = \{N\}$ and $BQ \cap CP = \{H\}$, prove that:

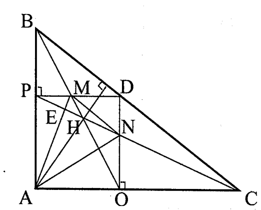
(a) $PM = DN$;
(b) $MN \parallel BC$;
(c) $AH \perp BC$.

Solution. (a) It is easy to see that $APDQ$ is a square. As triangles BPM and BAQ, CND, and CPB, CDQ and CBA are respectively similar, we have

$$\frac{PM}{AQ} = \frac{BP}{BA}, \tag{1}$$

and

$$\frac{DN}{BP} = \frac{CD}{CB} = \frac{DQ}{BA} \quad \text{or} \quad \frac{DN}{DQ} = \frac{BP}{BA}. \tag{2}$$



From (1), (2) and $AQ = DQ$ we obtain $PM = DN$.

(b) By Thales theorem we have

$$\frac{PN}{NC} = \frac{DN}{NQ}. \tag{3}$$

As $DN = PM$ and $NQ = DQ - DN = PD - PM = MD$, relation (3) reads

$$\frac{PN}{NC} = \frac{PM}{MD},$$

that is $MN \parallel BC$.

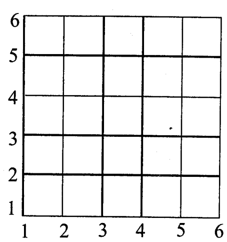
(c) Triangles APM and PDN are equal. Let E be the intersection of lines PC and AM. We have $\angle AEP = \angle EPM + \angle PME = \angle NPD + \angle PND = 90°$, that is $NH \perp AM$ and analogously $MH \perp AN$. Thus H is the orthocenter of the triangle AMN and $AH \perp MN$. From $MN \parallel BC$ the conclusion is obvious.

PROBLEM 3. A grid consists of $2n$ vertical and $2n$ horizontal lines, each group disposed at equal distances. The lines are all painted in red and black, such that exactly n vertical and n horizontal lines are red.

Find the smallest n such that for any painting satisfying the above condition, there is a square formed by the intersection of two vertical and two horizontal lines, all of the same colour.

Solution. It is clear that for $n = 1$ the vertical bars and the horizontal ones can be differently painted. For $n = 2$ we paint the extreme vertical bars in black and the middle two horizontal in red. We obtain thus a configuration with no square of the same colour.

We prove that the answer is $n = 3$.



Let $a < b < c$ the vertical bars and $x < y < z$ the horizontal ones painted in black. If in any of the three numbers sets there are two consecutive numbers, we are done.

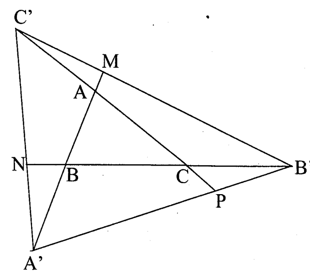
Suppose that this is not the case. Then the numbers $b-a, c-b$ and $c-a$ can be only $2, 2, 4$ or $2, 3, 5$. It will suffice to show that in any way we choose $x < y < z$ from the set $\{1, 2, 3, 4, 5, 6\}$, one of the numbers $y - z$, $z - y$, $z - x$ is 2 or one of the pairs is $3, 4$ or $4, 5$ which is obvious by writing down any case.

PROBLEM 4. Consider a triangle ABC. Let B' the reflection of B with respect to C, C' the reflection of C with respect to A and A' the reflection of A with respect to B.

(a) Prove that the area of $AC'A'$ is two times the area of ABC.

(b) If we erase the points A, B, C, is it possible to reconstruct them? Justify!

Solution. (a) Use the obvious fact that a median divides the triangle in two triangles of the same area. As BA and $C'B$ are medians in the triangles CBC' and $AC'A'$ we find $\sigma(ABC) = \sigma(ABC') = \sigma(BC'A')$ that is $\sigma(AC'A') = 2\sigma(ABC)$.



(b) Consider the points M, N, P which are respectively the intersections of lines AA', BB', CC' with $B'C', C'A', A'B'$. The parallel line to BC that contains A cuts $A'C'$ in D. It follows that AD is middle line in the triangle $CC'N$ and BN is middle line in the triangle $AA'D$, thus $C'D = DN = NA'$.

We conclude that, given the points A', B', C' we choose points N, M, P on the segments $C'A', B'C', A'B'$ respectively such that

$$\frac{A'N}{A'C'} = \frac{C'M}{C'B'} = \frac{B'P}{B'A'} = \frac{1}{3}.$$

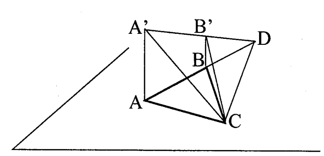
It is clear that A, B, C can be recovered as the points of intersection of pairs of lines $A'M$ and $C'N$, $A'M$ and $B'P$, $B'P$ and $C'N$ respectively.

8th GRADE

PROBLEM 1. Let ABC be an equilateral triangle. The perpendiculars AA' and BB' on the plane containing ABC at the points A and B are $AA' = AB$ and $BB' = \frac{1}{2}AB$.

Find the angle between the planes (ABC) and $(A'B'C')$.

Solution. Let D be the point of intersection of lines AB and $A'B'$.



It is easy to see that BB' is middle line in the triangle $AA'D$ and thus $AB = BD$. In the triangle CAD the median CB is half of AD, thus $\angle ACD = 90°$. By the Three perpendicular theorem we conclude $A'C \perp CD$ and $\angle A'CA = 45°$ is the angle made by the planes (ABC) and $(A'B'C)$.

PROBLEM 2. Let $M \subset \mathbf{R}$ be a finite set containing at least two elements. We say that the function f has property $\mathcal{P}$ if $f : M \to M$ and there are $a \in \mathbf{R}^*$ and $b \in \mathbf{R}$ such that $f(x) = ax + b$.

(a) Show that there is at least a function having property $\mathcal{P}$.

(b) Show that there are at most two functions having property $\mathcal{P}$.

(c) If M has 2003 elements with sum 0 and if there are two functions with property $\mathcal{P}$, prove that $0 \in M$.

Solution. (a) Let n be the cardinality of M and order its elements as $x_1 < x_2 < \cdots < x_n$. If $a > 0$ and $f(x) = ax + b$ has property $\mathcal{P}$, then $ax_1 + b < ax_2 + b < \cdots < ax_n + b$, thus $f(x_1) < f(x_2) < \cdots < f(x_n)$. As $f(x_i) \in M$ for any $i = 1, 2, \ldots, n$ we must have $f(x_i) = x_1$ for any $i = 1, 2$. Then $x_1(a-1) + b = x_2(a-1) + b = 0$, thus $(x_1 - x_2)(a-1) = 0$, implying $a = 1$ and $b = 0$. We conclude $f(x) = x$, for any $x \in M$ in this case.

If $a < 0$ we deduce $f(x_n) < f(x_{n-1}) < \cdots < f(x_1)$ which implies $f(x_n) = x_1$, $f(x_{n-1}) = x_2, \ldots, f(x_1) = x_n$. As in the preceeding case we must have $a(x_1 - x_n) = x_n - x_1$ which implies $a = -1$ and $b = x_n - ax_1 = x_1 + x_n$.

We conclude that there are at most two functions (the last case gives a function with property $\mathcal{P}$ if and only if $x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1}$).

(c) Let $x_1 < x_2 < \cdots < x_{2003}$ the elements of M having sum 0. The two functions that one can define are given by $f_1(x) = x$ and $f_2(x) = -x + b$ where $b = x_1 + x_{2003}$. We have from the preceeding remarks that $f_2(x_1) = x_{2003}$, $f_2(x_2) = x_{2002}, \ldots, f_2(x_{2003}) = x_1$, thus $f_2(x_1) + f_2(x_2) + \cdots + f_2(x_{2003}) = 0$. This implies $b = 0$ and in turn $f_2(x) = -x$. It follows $f_2(x_{1002}) = -x_{1002}$ and on the other side $f_2(x_{1002}) = x_{1002}$. We conclude $x_{1002} = 0 \in M$.

PROBLEM 3. Consider an array $n \times n$ $(n \geqslant 2)$ with n^2 integers. In how many ways one can complete the array if the product of the numbers on any row and column is 5 or -5?

Solution. The number of ways one can choose entries with ± 5 is $n!$. The $\pm$ signs of the 5-th and the $\pm$ 1's can then be chosen in 2^{n^2} ways. Thus the total number of array completions is $2^{n^2} n!$.

PROBLEM 4. (a) Let MNP be a triangle with $\angle MNP > 60°$. Prove that MP is not the smallest side of the triangle.

(b) A plane contains an equilateral triangle ABC. The point V, that doesn't belong to the plane (ABC) is such that $\angle VAB = \angle VBC = \angle VCA$. Prove that if $VA = AB$, then all sides of the piramid $VABC$ are equal.

Solution. (a) In a triangle the side opposing a smaller angle is smaller. As the sum of the angles in a triangle is 180°, the smallest angle cannot be larger than 60°. Thus, if $\angle MNP > 60°$, then MP is not the smallest side of the triangle.

(b) It will be enough to prove that $\angle VAB = \angle VBC = \angle VCA = 60°$. These will simply imply that $AB = VB = VA$ and $VA = VC$ as asked.

By way of contradiction, suppose $\angle VAB = \angle VBC = \angle VCA > 60°$. From (a) we must have $VB > VA = AB = BC$. In the triangle VBC we have $\angle VBC > 60°$, thus VC is not the smallest side. Then $VC > BC = CA$. In the triangle VAC we have $\angle VCA > 60°$, thus VA is not the smallest side. But $VA = CA < VC$, a contradiction. The case $\angle VAB = \angle VBC = \angle VCA < 60°$ can be treated in a similar manner using a): if $\angle MNP < 60°$, then MP is not the largest side of the triangle.

9th GRADE

PROBLEM 1. Find all functions $f : \mathbf{N}^* \to \mathbf{N}^*$ such that for any n, $n \geqslant 1$, the number

$$f(1) + f(2) + \cdots + f(n)$$

is the cube of a number at most equal to n.[1]

Solution. As the function f takes only positive values we have $f(1) < f(1) + f(2) < \cdots < f(1) + f(2) + \cdots + f(n)$, for any $n \geqslant 1$. The n listed numbers are by hypothesis elements of the set $\{1^3, 2^3, \ldots, n^3\}$. In conclusion $f(1) + f(2) + \cdots + f(3) = n^3$ for any $n \geqslant 1$. Thus $f(1) = 1$ and

$$\begin{aligned} f(n) &= (f(1) + f(2) + \cdots + f(n)) - (f(1) + f(2) + \cdots + f(n-1)) \\ &= n^3 - (n-1)^3 = 3n^2 - 3n + 1 \end{aligned}$$

for any $n \geqslant 1$.

PROBLEM 2. Find $n \in \mathbf{N}$, $n \geqslant 2$ and digits $a_1, a_2, \ldots, a_n$, such that

$$\sqrt{\overline{a_1 a_2 \ldots a_n}} - \sqrt{\overline{a_1 a_2 \ldots a_{n-1}}} = a_n.$$

[1] $\mathbf{N}^*$ is the set of positive integers

($\overline{a_1a_2\dots a_n}$ is the n-digit number with digits $a_1, a_2, \dots, a_n$).

Solution. Let $x = \overline{a_1a_2\dots a_{n-1}} \in \mathbf{N}$. We get $\overline{a_1a_2\dots a_n} = 10x + a_n$ and $\sqrt{10x + a_n} - \sqrt{x} = a_n$ or $10x + a_n = x + a_n^2 + 2a_n\sqrt{x}$.

The last relation rewrites $9x = a_n(a_n + 2\sqrt{x} - 1)$. As $a_n \leqslant 9$ we get $x \leqslant a_n + 2\sqrt{x} - 1$ or $(\sqrt{x} - 1)^2 \leqslant a_n \leqslant 9$ which implies $\sqrt{x} \leqslant 4$ or $x \leqslant 16$.

On the other side $a_n \neq 0$ (otherwise $x = 0$) and $\sqrt{x} = \frac{9x + a_n - a_n^2}{2a_n}$ is a rational number. Thus x must be a perfect square. As $\sqrt{10x + a_n} = a_n + \sqrt{x}$ we get that $10x + a_n$ is a perfect square. Considering the possible case $x = 1, 4, 9, 16$ we find $x = 16$, $a_n = 9$ and $n = 3$. It is easy to verify that $\sqrt{169} - \sqrt{16} = 9$.

PROBLEM 3. On the blackboard there are given points A, B, C, D. Vlad constructs the points A', B', C', D' in the following manner: A' is the reflexion of A with respect to B, B' is the reflexion of B with respect to C, C' is the reflexion of C with respect to D and D' is the reflexion of D with respect to A. Maria eliminates from the blackboard the points A, B, C, D.

Can Vlad reconstruct the positions of these points? Justify; vectors can be used.

Solution. Let $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ denote the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$ respectively. We have $\vec{a} + \vec{b} + \vec{c} + \vec{d} = 0$. Denote also by $\vec{x}$, $\vec{y}$, $\vec{z}$ the vectors $\overrightarrow{A'B'}$, $\overrightarrow{B'C'}$, $\overrightarrow{C'D'}$. We have

$$\begin{aligned}
\vec{x} &= \overrightarrow{A'B'} = \overrightarrow{A'B} + \overrightarrow{BB'} = -\vec{a} + 2\vec{b}\\
\vec{y} &= \overrightarrow{B'C'} = \overrightarrow{B'C} + \overrightarrow{CC'} = -\vec{b} + 2\vec{c}\\
\vec{z} &= \overrightarrow{C'D'} = \vec{c} + 2\vec{d} = -\vec{c} + 2(-\vec{a} - \vec{b} - \vec{c}) = -2\vec{a} - 2\vec{b} - 3\vec{c}).
\end{aligned}$$

It follows $\vec{b} = 2\vec{c} - \vec{y}$, $\vec{a} = 2\vec{b} - \vec{x} = 4\vec{c} - 2\vec{y} - \vec{x}$ and $\vec{z} = -8\vec{c} + 4\vec{y} + 2\vec{x} - 4\vec{c} + 2\vec{y} - 3\vec{c}$. Finally, $\vec{c} = \frac{1}{15}(6\vec{y} + 2\vec{x} - \vec{z})$.

To recover D we consider the vector $-\vec{c} = \overrightarrow{C'D}$, with C' as origin. Then A is the middle of DD', B the middle of AA' and C the middle of BB'.

PROBLEM 4. A set A of nonzero vectors in the plane have property (S) if it consists of at least three elements and for any $\vec{u} \in A$ there are vectors $\vec{v}, \vec{w} \in A$ such that $\vec{v} \neq \vec{w}$ and $\vec{u} = \vec{v} + \vec{w}$.

(a) Prove that, for any $n \geqslant 6$, there is a set of vectors having property (S).

(b) Prove that any finite set of vectors with property (S) has at least six elements.

Solution. (a) Proceed by induction on $n \geqslant 6$. For $n = 6$ consider a triangle ABC and the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{BA}$, $\overrightarrow{CB}$, $\overrightarrow{AC}$.

For $n > 6$ and A a set of n nonzero vectors $v_1, \dots, v_n$ with property (S), let v_i, v_j two different vectors from A which realize the minimum angle between vectors in A.

We have $v_i + v_j \notin A$, otherwise the minimality conditions is brocken. Thus the set $A \cup \{v_i + v_j\}$ has $n + 1$ vectors and satisfies property (S).

(b) Consider a configuration with all vectors having common origin O, thus $A = \{\overrightarrow{OX_1}, \dots, \overrightarrow{OX_n}\}$. Consider two non-parallel directions given by vectors $\vec{u}$ and $\vec{v}$, in such a way that $\vec{u}$ nor $\vec{v}$ are parallel with any of the vectors in A or with any of the vectors $\overrightarrow{X_iX_j}$, $i \neq j$.

Denote $\overrightarrow{OX_i} = a_i\vec{u} + b_i\vec{v}$ the decomposition of $\overrightarrow{OX_i}$ for $i = 1, \dots, n$. The set of real numbers $M = \{a_1, a_2, \dots, a_n\}$ has a property similar to (S). Let a be the maximum of M. It is clear that $a > 0$ and there are $b, c > 0$ such that $a = b + c$ and $b \neq c$. If not a cannot be the largest element of M.

In a similar way, for the minimum of M, say a', we can find $b', c' \in M$ with $b', c' < 0$ and $b' \neq c'$ such that $a' = b' + c'$. We have thus produced six different elements of M.

10th GRADE

PROBLEM 1. In the interior of a cube there are 2003 points. Prove that, one can divide the cube in more than 2003^3 smaller cubes, such that any of the given points is in the interior of a small cube (not on the borders).

Solution. There are many obvious ways in proving the result. The following is maybe the most elegant.

Consider a partition of the cube in n^3 equal small cubes. Take n as large such that all the small cubes which have a face in common with the faces of the given cube did not contain points in the interior or on the faces. Choose then n such that $n^2 > 2003^3$ to answer the condition is the problem for the small cubes discussed and the remaining cube.

PROBLEM 2. Determine all functions $f : \mathbf{N}^* \to M$ having the property that

$$1 + f(n)f(n+1) = 2n^2\left(f(n+1) - f(n)\right),$$

for any $n \in \mathbf{N}^*$, in any of the situations

(a) $M = \mathbf{N}$;

(b) $M = \mathbf{Q}$.

Solution. Let $f : \mathbf{N}^* \to \mathbf{R}$ be the given function. If $f(n+1)f(n) = -1$ then $f(n+1) = f(n)$ and $f^2(n) + 1 = 0$ which is not possible

We can thus write the recurrence relation as

$$\frac{f(n+1) - f(n)}{1 + f(n)f(n+1)} = \frac{1}{2n^2}.$$

Put $x_n = \arctan f(n)$ or $f(n) = \tan x_n$. The given relation rewrites

$$\tan(x_{n+1} - x_n) = \frac{1}{2n^2}$$

or

$$x_{n+1} - x_n = \arctan\frac{1}{2n^2} + p_n\pi$$

with $p_n \in \mathbf{Z}$. But

$$\sum_{k=1}^{n-1} \arctan\frac{1}{2k^2} = \sum_{k=1}^{n-1}(\arctan(2k+1) - \arctan(2k-1)) = \arctan(2n-1) - \frac{\pi}{4}.$$

Thus

$$\arctan(2n-1) - \frac{\pi}{4} = \sum_{k=1}^{n-1}((x_{n+1} - x_n) - p_k\pi) = x_n - x_1 - l_n\pi,$$

where $l_n \in \mathbf{Z}$. We obtain

$$x_n = \arctan(2n-1) + x_1 - \frac{\pi}{4} + l_n\pi$$

and as a consequence

$$f(n) = \tan x_n = \frac{2n - 1 + \tan\left(x_1 - \frac{\pi}{4}\right)}{1 - (2n-1)\tan\left(x_1 - \frac{\pi}{4}\right)} = \frac{n(a+1) - 1}{a - n(a-1)}$$

where $a = \tan x_1 = f(1)$. We must have

$$a \notin \left\{\frac{n}{n-1} \mid n \in \mathbf{N},\, n \geqslant 2\right\}.$$

(a) If $f : \mathbf{N}^* \to \mathbf{N}$, for $a \geqslant 2$ we have $f(3) = \frac{3a+2}{3-2a} < 0$, a contradiction. If $a = 0$ then $f(2) = \frac{1}{2} \notin \mathbf{N}$. Thus $a = 1$ and $f(n) = 2n - 1$ which verifies the given relation.

(b) Denote $f(1) = a \in \mathbf{Q}$, $a \notin \left\{\frac{n}{n-1} \mid n \in \mathbf{N},\, n \geqslant 2\right\}$. The function given by

$$f(n) = \frac{n(a+1) - 1}{a - n(a-1)}$$

verifies the given relation.

PROBLEM 3. Let ABC be a triangle.

(a) Prove that if M is any point in its plane, then

$$AM \sin A \leqslant BM \sin B + CM \sin C.$$

(b) Let A_1, B_1, C_1 be points on the sides BC, AC and AB respectively, such that the angles of the triangle $A_1B_1C_1$ are in this order α, β, γ. Prove that

$$\sum AA_1 \sin\alpha \leqslant \sum BC \sin\alpha.$$

Solution. (a) Consider a complex plane with origin in M. Denote by a, b, c the complex coordinates of A, B, C, respectively. As $a(b-c) = b(a-c) + c(b-a)$ we have $|a|\,|b-a| = |b(a-c) + c(b-a)| \leqslant |b|\,|a-c| + |c|\,|b-a|$. Thus $AM \cdot BC \leqslant BM \cdot AC + CM \cdot AB$ or $2R \cdot AM \cdot \sin A \leqslant 2R \cdot BM \cdot \sin B + 2R \cdot CM \cdot \sin C$ which gives $AM \cdot \sin A \leqslant BM \sin B + CM \cdot \sin C$.

(b) From (a) we have

$$AA_1 \sin\alpha \leqslant AB_1 \sin\beta + AC_1 \sin\gamma$$

$$BB_1 \sin\beta \leqslant BA_1 \sin\alpha + BC_1 \sin\gamma$$

$$CC_1 \sin\gamma \leqslant CA_1 \sin\alpha + CB_1 \sin\beta$$

which summed up give the desired conclusion.

PROBLEM 4. Given positive numbers a, b, c, d such that $a > c > d > b > 1$ and $ab > cd$, prove that the function $f : [0, \infty) \to \mathbf{R}$, $f(x) = a^x + b^x - c^x - d^x$, is strictly increasing.

Solution. Write

$$\begin{aligned} f(x) &= a^x + b^x - c^x - \left(\frac{ab}{c}\right)^x + d^x\left(\frac{ab}{cd}\right)^x - d^x \\ &= c^x\left[\left(\frac{a}{c}\right)^x - 1\right] + b^x\left[1 - \left(\frac{a}{c}\right)^x\right] + d^x\left[\left(\frac{ab}{cd}\right)^x - 1\right] \\ &= b^x\left[\left(\frac{a}{c}\right)^x - 1\right]\left[\left(\frac{c}{b}\right)^x - 1\right] + d^x\left[\left(\frac{ab}{cd}\right)^x - 1\right]. \end{aligned}$$

As $b > 1$, $\frac{a}{c} > 1$, $\frac{c}{b} > 1$, $d > 1$ and $\frac{ab}{cd} > 1$, all functions involved are increasing and positive, hence f is increasing.

11th GRADE

PROBLEM 1. In the Cartesian plane xOy consider the collinear points $A_i(x_i, y_i)$, $i = \overline{1,4}$, such that there are invertible matrices $M \in \mathcal{M}_4(\mathbf{C})$ having the first two rows (x_1, x_2, x_3, x_4) and respectively (y_1, y_2, y_3, y_4).

Prove that for such a matrix M the sum of elements of M^{-1} is independent of M.

Solution. Let $ax + by + c = 0$ be the equation of the line that contains the points A_i, $i = 1, 2, 3, 4$. Remark that $c \neq 0$; otherwise $ax_i + by_i = 0$ for $i = 1, 2, 3, 4$ implies $\det M = 0$.

Let $M^{-1} = (z_{hk})$, that is

$$\sum_{h=1}^{4} x_h z_{hk} = \delta_{1k}, \quad \text{and} \quad \sum_{h=1}^{4} y_h z_{hk} = \delta_{2k} \quad \text{for } k = 1, 2, 3, 4.$$

We deduce

$$\sum_{h=1}^{4} (ax_h + by_h) z_{hk} = a\delta_{1k} + b\delta_{2k} \quad \text{for } k = 1, 2, 3, 4.$$

Thus

$$\sum_{h=1}^{4} z_{hk} = -\frac{a\delta_{1k} + b\delta_{2k}}{c} \quad \text{for } k = 1, 2, 3, 4.$$

Summing over k we conclude that the sum of elements of M^{-1} is $-\frac{a+b}{c}$.

PROBLEM 2. Let $f : [0, 1] \to [0, 1]$ be a function that is continuous at the points 0 and 1, has limits from the left and from the right at any point and verifies $f(x - 0) \leqslant f(x) \leqslant f(x + 0)$ for any $x \in (0, 1)$.

Prove that there is $x_0 \in (0, 1)$ such that $f(x_0) = x_0$.

Solution. Denote $A = \{x \in [0, 1] \mid f(x) \geqslant x\}$. As $f(0) \geqslant 0$ we deduce $0 \in A$. Consider $x_0 = \sup A \in [0, 1]$. We have the following posibilities:

(i) $x_0 = 0$. Consider $x_n \in (0, 1]$, $\lim x_n = 0$. As $f(x_n) < x_n$ and f is continuous we get $f(0) \leqslant 0$ or $f(0) = 0$.

(ii) $x_0 = 1$. As above, let $x_n \in A$ with $\lim x_n = 1$. From $f(x_n) \geqslant x_n$ we deduce by continuity $f(1) \geqslant 1$ which together with $f(1) \leqslant 1$ reads $f(1) = 1$.

(iii) $x_0 \in (0, 1)$. If $x_n \in (x_0, 1)$ with $\lim x_n = x_0$ from $f(x_0, 0) \leqslant x_0$ and $f(x_n) < x_n$ we get $f(x_0) \leqslant x_0$.

Suppose $f(x_0) < x_0$. Then $x_0 \notin A$ and as $x_0 = \sup A$ we find $y_n \in A$, $\lim y_n = x_0$. But $y_n < x_0$ and $f(y_n) \geqslant y_n$ implies $f(x_0 - 0) \geqslant x_0$. In conclusion $f(x_0) \geqslant x_0$, a contradiction. Thus only $f(x_0) = x_0$ is possible.

PROBLEM 3. (a) Prove that any matrix $A \in \mathcal{M}_n(\mathbf{C})$ is the sum of n matrices of rank 1.

(b) Prove that I_n cannot be written as the sum of less than n matrices of rank 1.

Solution. (a) Let for $A \in \mathcal{M}_n(\mathbf{C})$. A_k denote the matrix obtained replacing by 0 all rows except the k^{th}. If any of the rows of A is nonzero, then $\operatorname{rank} A_k \neq 0$, $k = 1, \ldots, n$ and $A = \sum_{k=1}^{n} A_k$ is a possibility of writting A as in the problem.

If the matrix has p nonzero rows and $n - p$ zero ones consider

$$A = (n - p + 1)B + A_{k_2} + \cdots + A_{k_p}$$

where $B = \frac{1}{n-p+1} A_{k_1}$ and $k_1, \ldots, k_p$ correspond to the nonzero rows. For $A = 0_n$ the result is obvious.

(b) Suppose that $I_n = B_1 + \cdots + B_k$, where B_i, $i = 1, \ldots, k$, $k < n$ are rank 1 matrices. As $\operatorname{rank}(A + B) \leqslant \operatorname{rank} A + \operatorname{rank} B$ for any matrices A, B, we obtain $n = \operatorname{rank} I_n \leqslant \operatorname{rank}(B_1) + \cdots + \operatorname{rank}(B_k) = k < n$, a contradiction.

PROBLEM 4. Let $\alpha > 1$ and let $f : [\frac{1}{\alpha}, \alpha] \to [\frac{1}{\alpha}, \alpha]$ be a bijective function. Suppose that $f^{-1}(x) = \frac{1}{f(x)}$ for any $x \in [\frac{1}{\alpha}, \alpha]$. Prove that:

(a) f has at least a discontinuity point;

(b) if f is continuous at 1, than f has an infinity of discontinuity points;

(c) there is a function f verifying the given conditions and possesing only a finite number of discontinuity points.

Solution. (a) As $f \circ f = \text{id}$ we find that for any $x \in [\frac{1}{\alpha}, \alpha]$ we have $\frac{1}{f(f(x))} = x$ that is $f(f(x)) = \frac{1}{x}$.

If f is continuous then, being injective, it is strictly monotone. Thus $f \circ f$ is increasing but $\frac{1}{x}$ is decreasing, a contradiction.

(b) As $f(f(\frac{1}{x})) = x = f(f^{-1}(x))$ we get $f^{-1}(x) = f(\frac{1}{x})$ implying $f(x) \cdot f(\frac{1}{x}) = 1$ for any $x \in [\frac{1}{\alpha}, \alpha]$. We obtain $f^2(1) = 1$ and as $f(1) > 0$ we conclude $f(1) = 1$. If f has a finite number of points of discontinuity we can find $a, b \in \mathbf{R}$, $\frac{1}{\alpha} < a < 1 < b < \alpha$ such that f is continuous on $[a, b]$. As above, we obtain that f is monotone on $[a, b]$, say increasing. Suppose $f([a, b]) = [c, d]$. Thus $c < 1 < d$. Let $x_0 \in (\max(a, c), 1)$. From $f(x_0) f^{-1}(x_0) = 1$ and $f(x_0) < f(1) = 1$ we get $f^{-1}(x_0) = \frac{1}{f(x_0)} > 1 = f^{-1}(1)$, a contradiction with the fact that f^{-1} is increasing on $[c, b]$.

(c) Consider $f : [\frac{1}{3}, 3] \to [\frac{1}{3}, 3]$ defined by $f(\frac{1}{3}) = 2$, $f(\frac{1}{2}) = \frac{1}{3}$, $f(1) = 1$, $f(2) = 3$, $f(3) = \frac{1}{2}$ and $f(x) = \frac{x-1}{2}$ for $x \in (2, 3)$.

Put $f(x) = \frac{1}{f^{-1}(x)} = \frac{1}{2x+1}$ for $x \in (\frac{1}{2}, 1)$. As $f(\frac{1}{2}, 1) = (\frac{1}{3}, \frac{1}{2})$ and the inverse on $(\frac{1}{3}, \frac{1}{2})$ is $f^{-1}(x) = \frac{1-x}{2x}$ we get $f(x) = \frac{1}{f'(x)} = \frac{2x}{1-x}$ on $(\frac{1}{3}, \frac{1}{2})$. In the same way we are imposed to define $f(x) = \frac{1}{f^{-1}(x)} = \frac{2+x}{x}$ for $x \in (1, 2)$.

12th GRADE

PROBLEM 1. Let $(G, \cdot)$ be a finite group with unity e. The least positive integer with the property that $x^n = e$, for any $x \in G$, is called the *exponent* of the group G.

(a) For any prime p, $p \geqslant 3$, show that the multiplicative group G_p consisting of those matrices of the form

$$\begin{pmatrix} \hat{1} & \hat{a} & \hat{b} \\ \hat{0} & \hat{1} & \hat{c} \\ \hat{0} & \hat{0} & \hat{1} \end{pmatrix} \quad \text{with } \hat{a}, \hat{b}, \hat{c} \in \mathbf{Z}_p,$$

is noncommutative and has exponent p.

(b) Show that, if $(G, \circ)$ and $(H, \bullet)$ are finite groups with exponents m respectively n, than the group $(G \times H, *)$ defined by $(g, h) * (g', h') = (g \circ g', h \bullet h')$, for any $(g, h), (g', h') \in G \times H$, has the exponent equal to the least common multiple of m and n.

(c) Deduce that any natural number n, $n \geqslant 3$ is the exponent of some finite noncommutative group.

Solution. (a) Let

$$A = \begin{pmatrix} \hat{1} & \hat{0} & \hat{0} \\ \hat{0} & \hat{1} & \hat{1} \\ \hat{0} & \hat{0} & \hat{1} \end{pmatrix}, \quad B = \begin{pmatrix} \hat{1} & \hat{1} & \hat{0} \\ \hat{0} & \hat{1} & \hat{0} \\ \hat{0} & \hat{0} & \hat{1} \end{pmatrix}.$$

A short computation shows that $AB \neq BA$, thus G_p is noncommutative. Let $X = I_3 + C$ where $C = \begin{pmatrix} \hat{0} & \hat{0} & \hat{b} \\ \hat{0} & \hat{0} & \hat{c} \\ \hat{0} & \hat{0} & \hat{0} \end{pmatrix}$. We obtain $X^p = I_3 + pC + \frac{p(p-1)}{2}C^2$. As p is prime, $2 \mid p - 1$ and as $pY = 0_3$ for any Y in G_p we conclude $X^p = I_3$, thus the exponent of G is p.

(b) Denote by k the exponent of $G \times H$. If $e_1 \in G$, $e_2 \in H$ are the unities of the two groups, from $(g^k, h^k) = (g, h)^k = (e_1, e_2)$ for any $g \in G$ and $h \in H$ we get $g^k = e_1$, $h^k = e_2$.

Let c, r be positive integers such that $r < m$ and $k = mc + r$. We have $e_1 = g^k = (g^m)^c \cdot g^r = g^r$ for any $g \in G$, and as m is exponent of G, we have $r = 0$, thus $m \mid k$. In the same way $r \mid k$, concluding $[m, n] \mid k$. As $m \mid [m, n]$ and $n \mid [m, n]$ we deduce $k = [m, n]$.

c) Let D_4 the group with 8 elements of the isometries in the plane that keep invariant the vertices of a square. D_4 is not commutative and its exponent is 4.

If n is not a power of 2, let $p \geqslant 3$, $p \mid n$. Consider $G_p \times \mathbf{Z}_n$ with exponent n. If $n = 2^k$, $k \geqslant 3$ consider $D_4 \times \mathbf{Z}_n$.

PROBLEM 2. Consider two different continuous functions $f, g : [0, 1] \to (0, \infty)$, such that $\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x$.

Let $(x_n)_{n \geqslant 0}$ the sequence defined by $x_n = \int_0^1 \frac{(f(x))^{n+1}}{(g(x))^n}\,\mathrm{d}x$.

(a) Prove that $\lim\limits_{n\to\infty} x_n = \infty$.

(b) Show that the sequence $(x_n)_{n \geqslant 0}$ is increasing.

Solution. (a) Suppose $f \leqslant g$. The function $h = f - g$ is continuous, not identically zero and $h \leqslant 0$. Thus $\int_0^1 h(x)\,\mathrm{d}x < 0$, a contradiction. Thus, we can find $c \in [0, 1]$ with $f(c) > g(c)$ or $\frac{f(c)}{g(c)} > 1$.

As $\frac{f}{g}$ is continuous, let $[\alpha, \beta]$ be an interval where $\frac{f(x)}{g(x)} > \lambda > 1$. We have

$$\int_0^1 f(x)\left(\frac{f(x)}{g(x)}\right)^n \mathrm{d}x \geqslant \int_\alpha^\beta f(x)\left(\frac{f(x)}{g(x)}\right)^n \mathrm{d}x \geqslant \lambda^n \int_\alpha^\beta f(x)\,\mathrm{d}x.$$

As $\lambda > 1$ and $\int_\alpha^\beta f(x)\,\mathrm{d}x = 0$ we get $\lim x_n = \infty$.

(b) $x_{n+1} - x_n = \int_0^1 \frac{(f^{n+1}(x) - g^{n+1}(x))(f(x) - g(x))}{g^{n+1}(x)}\,\mathrm{d}x \geqslant 0$, as the difference $f^{n+1}(x) - g^{n+1}(x)$ has the same sign as $f(x) - g(x)$.

PROBLEM 3. Let K be a finite field in which the polynomial $X^2 - 5$ is irreducible in $K[X]$. Show that:

(a) $1 + 1 \neq 0$;

(b) for any $a \in K$, the polynomial $X^5 + a$ is reducible in $K[X]$.

Solution. (a) If $1 + 1 = 0$ then $5 = 1$ and $x^2 - 5 = x^2 - 1(x - 1)(x + 1)$, a contradiction.

(b) If $a = 0$ the result is obvious. For $a \neq 0$ consider $f : K^* \to K^*$, $f(x) = x^5$. A straightfull calculation in $K[X]$ gives that $f(x) = 1$ is equivalent to

$$(x - 1)(x^4 + x^3 + x^2 + x + 1) = 0$$

or

$$4^{-1}x^2(x - 1)[(2x + 2x^{-1} + 1)^2 - 5] = 0.$$

But $(2x+2x^{-1}+1)^2$ cannot be 5 because otherwise the polynomial x^2-5 should have a root in K, contradicting the irreducibility.

Thus $f(x)=1$ implies $x=1$ property that simply implies that f is injective, thus bijective as defined on a finite set. In conclusion for any $a\in K$, $a\neq 0$ there is a $b\in K$ such that $b^5=a$, proving thus that the polynomial x^5-a has a root which implies that it is reducible.

PROBLEM 4. Consider the continuous functions $f:[0,\infty)\to\mathbf{R}$ and $g:[0,1]\to\mathbf{R}$. If $\lim_{x\to\infty} f(x)=L\in\mathbf{R}$, prove that

$$\lim_{n\to\infty}\frac{1}{n}\int_0^n f(x)g\left(\frac{x}{n}\right)\mathrm{d}x = L\int_0^2 g(x)\,\mathrm{d}x.$$

Solution. We can suppose $L=1$ and let $h=f-1$. We have

$$\begin{aligned}\frac{1}{n}\int_0^n f(x)g\left(\frac{x}{n}\right)\mathrm{d}x &= \frac{1}{n}\int_0^n h(x)g\left(\frac{x}{n}\right)\mathrm{d}x+\frac{1}{n}\int_0^n g\left(\frac{x}{n}\right)\mathrm{d}x\\ &= \frac{1}{n}\int_0^n h(x)g\left(\frac{x}{n}\right)\mathrm{d}x+\int_0^1 g(x)\,\mathrm{d}x.\end{aligned}$$

Let $M=\max|g(x)|$. Then

$$\left|\frac{1}{n}\int_0^n h(x)g\left(\frac{x}{n}\right)\mathrm{d}x\right|\leqslant\frac{M}{n}\int_0^n|h(x)|\,\mathrm{d}x.$$

Denote by $H:[0,\infty)\to\mathbf{R}$ the antiderivative $H(x)=\int_0^x|h(t)|\,\mathrm{d}t$. By the Hospital rule, as $H'=|h|$ we get:

$$\lim_{n\to\infty}\frac{1}{n}\int_0^n|h(x)|\,\mathrm{d}x=\lim_{x\to\infty}\frac{H(x)}{x}=\lim_{x\to\infty}|h|(x)=0,$$

finishing then the proof.

II.3. FINAL ROUND

7th GRADE

PROBLEM 1. Find the maximum number of elements which can be chosen from the set $\{1,2,3,\ldots,2003\}$ such that the sum of any two chosen elements is not divisible by 3.

Solution. Consider the sets $A=\{3,6,9,\ldots,2001\}$, $B=\{1,4,7,\ldots,2002\}$, and $C=\{2,5,8,\ldots,2003\}$.

The choice of a set as stipulated allows to use at most one element from A and elements from either B or C. Since $|B|=|C|=668$, we can obtain at most 669 elements; an example is $B\cup\{3\}$.

PROBLEM 2. Compute the maximum area of a triangle having a median of length 1 and a median of length 2.

Solution. Let ABC be the triangle, G be its barycenter and $BG=\frac{2}{3}$, $CG=\frac{4}{3}$. It follows

$$\operatorname{area}(ABC)=3\cdot\operatorname{area}(BCG)\leqslant 3\cdot\frac{\frac{2}{3}\cdot\frac{4}{3}}{2}=\frac{4}{3},$$

the equality being obtained when $BG\perp CG$.

PROBLEM 3. For every positive integer n consider

$$A_n=\sqrt{49n^2+0,35n}\,.$$

(a) Find the first three digits after the decimal point for A_1.

(b) Prove that the first three digits after the decimal point A_n and A_1 are the same, for every n.

Solution. (a) $A_1=\sqrt{49,35}=7,024\ldots$

(b) We have to prove that $a_n+0,024\leqslant A_n<a_n+0,025$, where $a_n=\lfloor A_n\rfloor$. From $(7n)^2<49n^2+0,35n<(7n+1)^2$ it follows that $a_n=7n$, and the inequalities

$$(7n+0,024)^2<49n^2+0,35n<(7n+0,025)^2$$

can be easily checked.

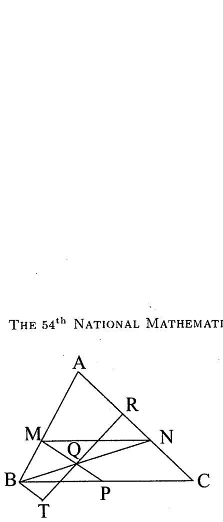
PROBLEM 4. In triangle ABC, P is the midpoint of side BC. Let $M\in(AB)$, $N\in(AC)$ be such that $MN\parallel BC$ and $\{Q\}$ be the common point of MP and BN. The perpendicular from Q on AC intersects AC in R and the parallel from B to AC in T. Prove that:

(a) $TP\parallel MR$;

(b) $\angle MRQ=\angle PRQ$.

Solution. (a) The conclusion says $\frac{QR}{QT}=\frac{QM}{QP}$.

This relation is obtained from $\frac{QR}{QT}=\frac{QN}{QB}$ (using similar triangles QRN and QTB) and from $\frac{QM}{QP}=\frac{QN}{QB}$ (in similar triangles QMN and QPB).



(b) Let $\{S\} = TP \cap AC$; $TBSC$ is a parallelogram. In the right triangle TRS, the median $[RP]$ is equal to PS, whence $\angle PRS \equiv \angle PSR \equiv \angle MRA$. It follows $m(\angle MRQ) = 90° - m(\angle MRA) = 90° - m(\angle PRS) = m(\angle PRQ)$.

8^{th} GRADE

PROBLEM 1. Let m, n be positive integers. Prove that the number $5^n + 5^m$ can be represented as sum of two perfect squares if and only if $n - m$ is even.

Solution. If m and n are both even then $m = 2k$, $n = 2l$ and $5^{2k} + 5^{2l} = (5^k)^2 + (5^l)^2$.

If m and n are both odd then $m = 2k+1$, $n = 2l+1$ and $5^{2k+1} + 5^{2l+1} = (5^k + 2 \cdot 5^l)^2 + (5^l - 2 \cdot 5^k)^2$.

For the converse, notice that if m and n have different parity then $5^m + 5^n = 5^{2k+1} + 5^{2l} \equiv 6 (\bmod 8)$ and, since the residues of the squares $(\bmod 8)$ are $0, 1, 4$, the sum of two squares cannot be $6 (\bmod 8)$.

PROBLEM 2. In a meeting there are 6 participants. It is known that among them there are seven pairs of friends and in any group of three persons there are at least two friends. Prove that:

(a) there exists a person who has at least three friends;

(b) there exists three persons who are friends with each other.

Solution. (a) If every person has at most 2 friends then the maximum number of pairs of friends would be $\frac{6 \cdot 2}{2} = 6$.

(b) Suppose A is friend with B, C and D. Since among $\{B, C, D\}$ there are at least two friends X, Y, the group $\{A, X, Y\}$ fulfils the requirements.

PROBLEM 3. The real numbers a, b fulfil the conditions

(i) $0 < a < a + \frac{1}{2} \leqslant b$;

(ii) $a^{40} + b^{40} = 1$.

Prove that b has the first 12 digits after the decimal point equal to 9.

Solution. Cearly $0 < a < b < 1$, whence $b > b^{40} = 1 - a^{40}$.

Therefore, in order to prove the conclusion it is enough to show that $a^{40} < \frac{1}{10^{12}}$, that is $a^{10} < \frac{1}{1000}$. This follows from $a \leqslant b - \frac{1}{2} < \frac{1}{2}$, whence $a^{10} < \frac{1}{2^{10}} < \frac{1}{1024}$.

PROBLEM 4. In tetrahedron $ABCD$, G_1, G_2 and G_3 are barycenters of the faces ACD, ABD and BCD respectively.

(a) Prove that the straight lines BG_1, CG_2 and AG_3 are concurent.

(b) Knowing that $AG_3 = 8$, $BG_1 = 12$ and $CG_2 = 20$ compute the maximum possible value of the volume of $ABCD$.

Solution. (a) Let M be the midpoint of $[CD]$. Using triangle ABM we get $G_1G_2 \parallel AB$ and $[AG_2]$, $[BG_1]$ have a common point G such that $\frac{BG}{GG_1} = \frac{AB}{G_1G_2} = \frac{BM}{G_2M} = 3$. In the same way CG_3 intersects $[BG_1]$ in a point G' such that $\frac{BG'}{G'G_1} = 3$. This shows that $G = G'$; also $\frac{BG}{GG_1} = \frac{AG}{GG_2} = \frac{CG}{GG_3} = 3$.

(b) From (a) we get $GA = 6$, $GB = 9$, $GC = 15$. Therefore,

$$\operatorname{vol}(ABCD) = 4 \operatorname{vol}(ABCG) \leqslant 4 \cdot \frac{GA \cdot GB \cdot GC}{6} = 4 \cdot \frac{6 \cdot 9 \cdot 15}{6} = 540,$$

and the equality is obtained when $GA \perp GB \perp GC \perp GA$.

9^{th} GRADE

PROBLEM 1. Find positive integers a, b if for every $x, y \in [a, b]$, $\frac{1}{x} + \frac{1}{y} \in [a, b]$.

Solution. From $x = y = b$ we get $\frac{2}{b} \geqslant a$, hence $ab \leqslant 2$, and $x = y = a$ leads to $\frac{2}{a} \leqslant b$, therefore $ab \geqslant 2$. It follows $ab = 2$, whence $a = 1$, $b = 2$. Also $x, y \in [1, 2]$ implies $\frac{1}{2} + \frac{1}{2} \leqslant \frac{1}{x} + \frac{1}{y} \leqslant \frac{1}{1} + \frac{1}{1}$, which shows that $a = 1$, $b = 2$ is a solution.

PROBLEM 2. An integer n, $n \geqslant 2$ is called *friendly* if there exists a family $A_1, A_2, \ldots, A_n$ of subsets of the set $\{1, 2, \ldots, n\}$ such that:

(1) $i \notin A_i$ for every $i = \overline{1, n}$;

(2) $i \in A_j$ if and only if $j \notin A_i$, for every distinct $i, j \in \{1, 2, \ldots, n\}$;

(3) $A_i \cap A_j$ is non-empty, for every $i, j \in \{1, 2, \ldots, n\}$.

Prove that:

(a) 7 is a friendly number;

(b) n is friendly if and only if $n \geqslant 7$.

Solution. (a) For $n = 7$ we can take

$$A_1 = \{2, 3, 4\}, \quad A_2 = \{3, 5, 6\}, \quad A_3 = \{4, 5, 7\}, \quad A_4 = \{2, 6, 7\}$$

$$A_5 = \{1,4,6\}, \quad A_6 = \{1,3,7\}, \quad A_7 = \{1,2,5\}.$$

(b) We can prove that every $n \geqslant 7$ is friendly using induction on n. The starting step has already been proven.

Suppose now that for some $n \geqslant 7$ there exists a family of sets $A_1, \ldots, A_n$ which shows that n is friendly. Then the family $B_1 = A_1, \ldots, B_n = A_n$ and $B_{n+1} = \{1, 2, \ldots, n\}$ proves that $n+1$ is also friendly.

It remains now to prove that every friendly number n is at least 7. We firstly notice that if $A_1, A_2, \ldots, A_n$ is a family of sets which shows that n is friendly then each A_i has at least 3 elements. Indeed, if $A_i \subset \{j, k\}$ then $A_i \cap A_j = \{k\}$ and $A_i \cap A_k = \{j\}$, whence $k \in A_j$ and $j \in A_k$, contradiction.

Consider now a $n \times n$ table with elements 0 or 1, where the element in row i and column j is 1 if and only if $j \in A_i$. This table has 0 on the principal diagonal and, for the remaining elements, the number of zeros equals the number of ones for ($i \neq j$ we have $a_{ij} = 0$ if and only if $a_{ij} = 1$). Hence the sum of the table's elements is $(n^2 - n)/2$. Since the sum of the elements in each row is at least 3, it follows $n^2 - n \geqslant 6n$, whence $n \geqslant 7$.

PROBLEM 3. Prove that the midpoints of the altitudes of a triangle are collinear if and only if the triangle is right.

Solution. In triangle ABC, the midpoints A', B', C' of the altitudes belong to the sides (produced if needed) of the median triangle MNP.

If triangle ABC is acute then A', B', C' are on the sides of MNP, therefore they cannot be collinear.

If triangle ABC is obtuse then two of A', B', C' are on the productions of two sides of MNP and the third is on the third side, that is they are not collinear.

Finally, if ABC is right-angled then A', B', C' are on the straight line drawn between the midpoints of the legs.

PROBLEM 4. Let P be a plane. Prove that there exists no function $f : P \to P$ such that for every convex quadrilateral $ABCD$, the points $f(A), f(B), f(C), f(D)$ are the vertices of a concave quadrilateral.

Solution. Suppose that such a function exists. Consider a convex pentagon $ABCDE$ and let $f(A) = A'$, $f(B) = B'$, $f(C) = C'$, $f(D) = D'$, $f(E) = E'$. The quadrilateral $A'B'C'D'$ is concave; without loss of generality we can assume that $D' \in \operatorname{int} A'B'C'$. Since A', B', C', E' are the vertices of a concave quadrilateral, it follows that $E' \in \operatorname{int} A'B'C'$ or, for instance, $A' \in \operatorname{int} B'C'E'$.

In the first case one of the quadrilaterals with vertices (A', B', D', E'), (B', C', D', E') or (A', C', D', E') is convex and in the second case (B', D', A', E') or (C', D', A', E') lead to a convex quadrilateral.

10th GRADE

PROBLEM 1. Let $OABC$ be a tetrahedron such that $OA \perp OB \perp OC \perp OA$, r be the radius of its inscribed sphere and H be the orthocenter of triangle ABC. Prove that $OH \leqslant r(\sqrt{3} + 1)$.

Solution. From $OC \perp OA$ and $OC \perp OB$ we get $OC \perp (OAB)$, hence $OC \perp AB$. From $CH \perp AB$ it follows $(OCH) \perp AB$, therefore $OH \perp AB$. In the same way $OH \perp AC$, whence $OH \perp (ABC)$. Denote now $OA = a$, $OB = b$, $OC = c$. Then

$$\begin{aligned} \operatorname{area}(ABC) &= \frac{1}{4}\sqrt{2\sum AB^2 \cdot AC^2 - \sum AB^4} \\ &= \frac{1}{4}\sqrt{2\sum (a^2+b^2)(a^2+c^2) - \sum (a^2+b^2)^2} = \frac{1}{2}\sqrt{\sum a^2 b^2}. \end{aligned}$$

From $3\operatorname{vol}(ABCO) = OH \cdot \operatorname{area}(ABC) = r(\operatorname{area}(AOB) + \operatorname{area}(AOC) + \operatorname{area}(BOC) + \operatorname{area}(ABC))$ we get $OH \cdot \sum a^2b^2 = r\left(\sum ab + \sqrt{\sum a^2b^2}\right)$ and we have to prove that $\sum ab \leqslant \sqrt{3\sum a^2b^2}$, which is (almost) obvious.

PROBLEM 2. The complex numbers z_i, $i = 1, \ldots, 5$, have the same non-nil modulus and $\sum\limits_{i=1}^{5} z_i = \sum\limits_{i=1}^{5} z_i^2 = 0$.

Prove that $z_1, z_2, \ldots, z_5$ are the complex coordinates of the vertices of a regular pentagon.

Solution. Consider the polynomial $P(X) = X^5 + aX^4 + bX^3 + cX^2 + dX + e$ with roots z_k, $k = \overline{1,5}$. Then $a = -\sum z_1 = 0$ and $b = \sum z_1 z_2 = \frac{1}{2}\left(\sum z_1\right)^2 - \frac{1}{2}\sum z_1^2 = 0$. Denoting by r the common modulus and taking conjugates we get also $0 = \sum \overline{z}_1 = \sum \frac{r^2}{z_1} = \frac{r^2}{z_1 z_2 z_3 z_4 z_5} \sum z_1 z_2 z_3 z_4$, whence $d = 0$ and $0 = \sum \overline{z}_1 \overline{z}_2 = \sum \frac{r^4}{z_1 z_2} = \frac{r^4}{z_1 z_2 z_3 z_4 z_5} \sum z_1 z_2 z_3$, therefore $c = 0$. It follows $P(X) = X^5 + e$, so $z_1, z_2, \ldots, z_5$ are the fifth roots of e and the conclusion is proved.

PROBLEM 3. Let a, b, c be the complex coordinates of the vertices A, B, C of a triangle. It is known that $|a| = |b| = |c| = 1$ and that there exists $\alpha \in (0, \frac{\pi}{2})$ such that $a + b\cos\alpha + c\sin\alpha = 0$. Prove that $1 < \operatorname{area}(ABC) \leqslant \frac{1+\sqrt{2}}{2}$.

Solution. Taking moduli we get, using $b\overline{b} = c\overline{c} = 1$, $1 = |a^2| = |b\cos\alpha + c\sin\alpha|^2 = (b\cos\alpha + c\sin\alpha)(\overline{b}\cos\alpha + \overline{c}\sin\alpha) = 1 + \sin\alpha\cos\alpha\left(\frac{b}{c} + \frac{c}{b}\right)$.

This shows that $b^2 + c^2 = 0$, whence $b = \pm ci$, so $m(\angle BOC) = 90°$.

Noticing now that the vector $\overline{OM} := \cos\alpha \cdot \overline{OB} + \sin\alpha \cdot \overline{OC}$ correspond to a point M belonging to the small arc $\overset{\frown}{BC}$ of the unit circle, we see that A is the reflection of M into O. This proves that the altitude from A of triangle ABC is larger then $\sqrt{2}$ but at most $1+\frac{\sqrt{2}}{2}$, therefore

$$\frac{1}{2}\cdot\sqrt{2}\cdot\sqrt{2} < \text{area}(ABC) \leqslant \frac{1}{2}\Big(1+\frac{\sqrt{2}}{2}\Big)\sqrt{2} = \frac{1+\sqrt{2}}{2}.$$

PROBLEM 4. A finite set A of complex numbers has the property: $z \in A$ implies $z^n \in A$ for every positive integer n.

(a) Prove that $\sum\limits_{z\in A} z$ is an integer.

(b) Prove that for every integer k one can choose a set A which fulfils the above condition and $\sum\limits_{z\in A} z = k$.

Solution. We will denote by $S(X)$ the sum of the elements of a finite set X. Suppose $0 \neq z \in A$. Since A is finite, there exists positive integers $m < n$ such that $z^m = z^n$, whence $z^{n-m} = 1$. Let d be the smallest positive integer k such that $z^k \in 1$. Then $1, z, z^2, \ldots, z^{d-1}$ are different and their d-th power is equal to 1, therefore these numbers are the d-th roots of the unity. This shows that $A \setminus \{0\} = \bigcup\limits_{k=1}^{m} U_{n_k}$, where $U_p = \{z \in \mathbf{C} \mid z^p = 1\}$. Since $S(U_p) = 0$ for $p \geqslant 2$, $S(U_1) = 1$ and $U_p \cap U_q = U_{(p,q)}$ we get

$$S(A) = \sum_k S(U_{n_k}) - \sum_{k<l} S(U_{n_k}\cap U_{n_l}) + \sum_{k<l<s} S(U_{n_k}\cap U_{n_l}\cap U_{n_s}) + \cdots = \text{integer}.$$

(b) Suppose that for some integer k there exists $A = \bigcup\limits_{k=1}^{m} U_{n_k}$ such that $S(A) = k$. Let $p_1, p_2, \ldots, p_6$ be the distinct primes which are not divisors of any n_k. Then

$$S(A\cup U_{p_1}) = S(A) + S(U_{p_1}) - S(A\cap U_{p_1}) = k - S(U_1) = k-1,$$

Also

$$\begin{aligned}
&S(A\cup U_{p_1p_2p_3}\cup U_{p_1p_4p_5}\cup U_{p_2p_4p_6}\cup U_{p_3p_5p_6})\\
&= S(A) + S(U_{p_1p_2p_3}) + S(U_{p_1p_4p_5}) + S(U_{p_2p_4p_6}) + S(U_{p_3p_5p_6})\\
&\quad - S(A\cap U_{p_1p_2p_3}) - \cdots + S(A\cap U_{p_1p_2p_3}\cap U_{P_1p_4p_5}) + \cdots\\
&\quad - S(A\cap U_{p_1p_2p_3}\cap U_{p_1p_4p_5}\cap U_{p_2p_4p_6}\cap U_{p_3p_5p_6})\\
&= k + 4\cdot 0 - 4\cdot S(U_1) - \sum_{k=1}^{6} S(U_{p_k})\\
&\quad + 10S(U_1) - 5S(U_1) + S(U_1) = k - 4 + 10 - 5 + 1 = k+2.
\end{aligned}$$

Hence, if there exists A such that $S(A) = k$ then there exist B and C such that $S(B) = k-1$ and $S(C) = k+2$; the conclusion follows now easily.

11th GRADE

PROBLEM 1. Find the locus of the points M from the plane of a rhombus $ABCD$ such that

$$MA\cdot MC + MB\cdot MD = AB^2.$$

Solution. Take coordinates such that $A(a,0)$, $B(0,b)$, $C(-a,0)$, $D(0,-b)$, and let $M(x,y)$. Then

$$\sqrt{(x-a)^2+y^2}\cdot\sqrt{(x+a)^2+y^2} + \sqrt{x^2+(y-b)^2}\cdot\sqrt{x^2+(y+b)^2} = a^2+b^2.$$

Using Cauchy's inequality we get

$$\begin{aligned}
&\sqrt{(x-a)^2+y^2}\cdot\sqrt{(x+a)^2+y^2} + \sqrt{x^2+(y-b)^2}\cdot\sqrt{x^2+(y+b)^2}\\
&\qquad \geqslant (a-x)(a+x) + y^2 + x^2 + (b-y)(b+y) = a^2+b^2\,.
\end{aligned}$$

Therefore, if M belongs to the locus then $(a-x)y = (a+x)y$, that is $xy = 0$.

If $y = 0$, the equation of the locus reduces to $a^2 + y^2 + |y^2 - b^2| = a^2 + b^2$, which is equivalent to $y \in [-b, b]$.

In the same way, $y = 0$ implies $x \in [-a, a]$.

Finally, the locus is $[AC]\cup[BD]$.

PROBLEM 2. Consider the real numbers $1 \leqslant a_1 < a_2 < a_3 < a_4$, $x_1 < x_2 < x_3 < x_4$ and the matrix $M = (a_i^{x_j})_{i,j\in\overline{1,4}}$.

Prove that $\det M > 0$.

Solution. We will prove inductively that $\det M_k > 0$ for $k \in \{2,3,4\}$, where M_k is obtained from M by taking the first k rows and columns. Denote $b_i = \frac{a_{i+1}}{a_1}$, whence $1 < b_1 < b_2 < b_3$.

For $k = 2$,

$$\det M_2 = a_1^{x_1+x_2}\begin{vmatrix} 1 & 1\\ b_1^{x_1} & b_2^{x_2}\end{vmatrix} > 0.$$

For $k = 3$,

$$\det M_3 = a_1^{x_1+x_2+x_3}\begin{vmatrix} 1 & 1 & 1\\ b_1^{x_1} & b_1^{x_2} & b_1^{x_3}\\ b_2^{x_1} & b_2^{x_2} & b_2^{x_3}\end{vmatrix} = a_1^{x_1+x_2+x_3} D(x_1,x_2,x_3).$$

Consider the function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = D(x_1, x_2, x)$. Its derivative is

$$f'(x) = b_2^x \ln b_2 \begin{vmatrix} 1 & 1 \\ b_1^{x_1} & b_1^{x_2} \end{vmatrix} - b_1^x \ln b_1 \begin{vmatrix} 1 & 1 \\ b_2^{x_1} & b_2^{x_2} \end{vmatrix} = \alpha b_2^x - \beta b_1^x,$$

with $\alpha, \beta > 0$, has exactly one root r and is positive for $x > r$ (because $b_2 > b_1$). Since $f(x_1) = f(x_2) = 0$ it follows that $r \in (x_1, x_2)$ and, because f is strictly increasing on (r, ∞), $x > x_2 \Rightarrow f(x) > f(x_2) = 0$.

For $k = 4$, as above

$$\det M_4 = a_1^{x_1+x_2+x_3+x_4} \begin{vmatrix} 1 & 1 & 1 & 1 \\ b_1^{x_1} & b_1^{x_2} & b_1^{x_3} & b_1^{x_4} \\ b_2^{x_1} & b_2^{x_2} & b_2^{x_3} & b_2^{x_4} \\ b_3^{x_1} & b_3^{x_2} & b_3^{x_3} & b_3^{x_4} \end{vmatrix} = a_1^{x_1+x_2+x_3+x_4} D(x_1, x_2, x_3, x_4).$$

The function $g(x) = D(x_1, x_2, x_3, x)$ has derivative of the form

$$g'(x) = \alpha b_3^x - \beta b_2^x + \gamma b_1^x = b_1^x \cdot \left(\alpha \left(\frac{b_3}{b_1} \right)^x - \beta \left(\frac{b_2}{b_1} \right)^x + \gamma \right) = b_1^x h(x)$$

and $h'(x) = 0$ has exactly one root. This show that $g'(x) = 0$ has at most two roots r_1, r_2; since $g(x_1) = g(x_2) = g(x_3) = 0$ it follows that $x_1 < r_1 < x_2 < r_2 < x_3$.

The conclusion comes now from the fact that g is a continuous function which does not vanish for $x > x_3$ and $\lim\limits_{x \to \infty} g(x) = \infty$.

PROBLEM 3. The real functions f, g are such that f is continuous and g is increasing and unbounded. It is known that for every sequence (x_n) of rational numbers with $(x_n)_n \to \infty$, $\lim\limits_{n \to \infty} f(x_n) g(x_n) = 1$.

Prove that $\lim\limits_{x \to \infty} f(x) g(x) = 1$.

Solution. We firstly notice that there exists M_0 such that $g(x) > 0$ for $x > M_0$ and there exists $M > M_0$ such that $f(x) \geqslant 0$ for $x > M_0$ (if this is not the case, then there exists a real sequence $(y_n)_n \to \infty$, $y_n > M_0$, such that $f(y_n) < 0$ and, taking for each y_n a rational z_n such that $y_n - 1 < z_n < y_n + 1$ and $f(z_n) < 0$ — this is possible because f is continous in y_n — we get a rational sequence $(z_n)_n \to \infty$ such that $f(z_n) \cdot g(z_n) < 0$).

We will now prove that for every $\varepsilon > 0$ there exists $\delta > M$ such that for every rational $x > \delta$, $|f(x)g(x) - 1| < \varepsilon$. Indeed, if for each $\delta_n := n$ there exist some rational $x_n > \delta_n$ such that $|f(x_n)g(x_n) - 1| \geqslant \varepsilon$ then $(x_n)_n$ would be a rational sequence such that $(x_n)_n \to \infty$ and $f(x_n)g(x_n) \not\to 1$, contradiction.

Take now $\varepsilon > 0$ and $\delta > M$ such that for every rational $x > \delta$, $|f(x)g(x) - 1| < \frac{\varepsilon}{2}$. Consider a real $x_0 > \delta$ and choose $x_1 > x_0$. Since $\lim\limits_{x \to x_0} f(x) = f(x_0)$, there exists rationals $a, b > \delta$ such that $a < x_0 < b$ and $|f(a) - f(x_0)| < \frac{\varepsilon}{2g(x_0)}$, $|f(b) - f(x_0)| < \frac{\varepsilon}{2g(x_1)}$.

It follows

$$f(x_0)g(x_0) > f(x_0)g(a) > \left(f(a) - \frac{\varepsilon}{2g(x_0)} \right) g(a) \geqslant f(a)g(a) - \frac{\varepsilon}{2} > 1 - \varepsilon$$

and

$$f(x_0)g(x_0) < f(x_0)g(b) < \left(f(b) + \frac{\varepsilon}{2g(x_1)} \right) g(b) \leqslant f(b)g(b) + \frac{\varepsilon}{2} < 1 + \varepsilon.$$

This shows that for an arbitrary real $x_0 > \delta$, $|f(x_0)g(x_0) - 1| < \varepsilon$ and ends the proof.

PROBLEM 4. Let A be a 3×3 matrix with real entries. Prove that:

(a) if f is a real polynomial without real roots then $f(A) \neq 0_3$;

(b) there exists a positive integer n such that

$$(A + A^*)^{2n} = A^{2n} + (A^*)^{2n}$$

if and only if $\det A = 0$.

Solution. (a) If f has no real roots then $f(x)$ has a constant sign; we can assume that $f(x) > 0$ for every $x \in \mathbf{R}$; since $\lim\limits_{x \to \pm\infty} f(x) = \infty$ it follows that the continuous function f has a minimum $m > 0$. The polynomial $g(X) = f(X) - \frac{m}{2}$ has no real roots, therefore its irreducible real factors are of the form $X^2 + 2aX + a^2 + b^2$. We notice now that for real a, b

$$\det(A^2 + 2aA + (a^2 + b^2)I_3) = \det(A + aI_3 + biI_3) \cdot \det(A + aI_3 - biI_3) \geqslant 0$$

as a product of two complex conjugate numbers, and $\det(-\frac{m}{2} I_3) = -\frac{m^3}{8} < 0$, therefore the equality $g(A) = -\frac{m}{2} I_3$ is impossible.

(b) If $\det A = 0$ then $A \cdot A^* = A^* \cdot A = 0_3$, hence $(A + A^*)^2 = A^2 + (A^*)^2$.

For the converse, if $(A + A^*)^{2n} = A^{2n} + (A^*)^{2n}$ and $d = \det A \neq 0$, then multiplying with A^{2n} we get $(A^2 + dI_3)^{2n} = A^{4n} + d^{2n} I_3$, whence $f(A) = 0_3$, where $f(X) = (X^2 + d)^{2n} - X^{4n} - d^{2n}$. Since $f'(X) = 4nX[(X^2 + d)^{2n-1} - X^{4n-2}]$ has only the simple root 0, it follows that 0 is the only root of f, that is $f(X) = dX^2 \cdot g(X)$, where g is a polynomial without real roots. From $f(A) = 0_3$ we get now $dA^2 \cdot g(A) = 0_3$, whence $g(A) = \frac{1}{d} 0_3 \cdot A^{-1} = 0_3$, which contradicts a).

12th GRADE

PROBLEM 1. (a) Let K be a field and $n \geqslant 2$ be an integer. Describe the set

$$Z(\mathcal{M}_n(K)) = \{A \in \mathcal{M}_n(K) \mid AX = XA \text{ for every } X \in \mathcal{M}_n(K)\}$$

and prove that the ring $Z(\mathcal{M}_n(K))$ is isomorphic to K.

(b) Prove that the rings $\mathcal{M}_n(\mathbf{R})$ and $\mathcal{M}_n(\mathbf{C})$ are not isomorphic.

Solution. (a) Let $E_{1i} = (e_{lk})_{1 \leqslant k,l \leqslant n}$, where $e_{kl} = 1$ for $k = 1$, $l = i$ and $e_{kl} = 0$ otherwise, and $A = (a_{kl})_{1 \leqslant k,l \leqslant n} \in Z(\mathcal{M}_n(K))$. Then $AE_{1i} = E_{1i}A$, leads to $a_{i1} = a_{i2} = \cdots = a_{i\,i-1} = a_{i\,i+1} = \cdots = a_{in} = 0$ and $a_{11} = a_{ii}$. This shows that A is of the form aI_n, $a \in K$; for such A we have indeed $AX = XA = aX$ for every $X \in \mathcal{M}_n(K)$.

Moreover, the function $f : K \to Z(\mathcal{M}_n(K))$, $f(a) = aI_n$ is an isomorphism from the field K to the ring $Z(\mathcal{M}_n(K))$.

(b) Suppose there exists an isomorphism $f : \mathcal{M}_n(\mathbf{R}) \to \mathcal{M}_n(\mathbf{C})$. Then it follows easily that $f(Z(\mathcal{M}_n(\mathbf{R}))) = Z(\mathcal{M}_n(\mathbf{C}))$; using (a) we obtain that the fields $\mathbf{R}$ and $\mathbf{C}$ must be isomorphic. But for an isomorphism $g : \mathbf{R} \to \mathbf{C}$ and $a = g^{-1}(\mathrm{i}) \in \mathbf{R}$ we would get $a^2 = g^{-1}(\mathrm{i}^2) = g^{-1}(-1) = -1$, contradiction.

PROBLEM 2. Let $n \geqslant 3$ be an odd integer. Find all continuous functions $f : [0,1] \to \mathbf{R}$ such that

$$\int_0^1 \left(f\left(\sqrt[k]{x}\right)\right)^{n-k} \mathrm{d}x = \frac{k}{n},$$

for every $k \in \{1, \ldots, n-1\}$.

Solution. The substition $x = t^k$ leads to

$$\int_0^1 (f(t))^{n-k} \cdot t^{k-1}\,\mathrm{d}t = \frac{1}{n},$$

for every $k \in \{1, 2, \ldots, n-1\}$ and also for $k = n$. Therefore

$$\sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \int_0^1 (f(t))^{n-k}\, t^{k-1}\,\mathrm{d}t = \frac{1}{n} \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} = \frac{1}{n}(1-1)^{n-1} = 0,$$

which leads to $\int_0^1 (f(t) - t)^{n-1}\,\mathrm{d}t = 0$. From $n - 1 = \text{even} > 0$ and $f - \mathrm{Id}_{[0,1]}$ continuous we obtain $f = \mathrm{Id}_{[0,1]}$.

PROBLEM 3. A continuous function $f : \mathbf{R} \to \mathbf{R}$ fulfils the condition $xf(x) \geqslant \int_0^x f(t)\,\mathrm{d}t$, for every real x.

(a) Prove that the function $g : \mathbf{R}^* \to \mathbf{R}$, $g(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t$ is increasing on $(-\infty, 0)$ and on $(0, \infty)$.

(b) Prove that if f has also the property

$$\int_x^{x+1} f(t)\,\mathrm{d}t = \int_{x-1}^{x} f(t)\,\mathrm{d}t \quad \text{for all real } x$$

then f is constant.

Solution. Let $F(x) = \int_0^x f(t)\,\mathrm{d}t$.

(a) We have $\left(\frac{F(x)}{x}\right)' = \frac{xf(x) - F(x)}{x^2} \geqslant 0$, for every $x \neq 0$.

(b) The given property says that $F(x+1) - F(x) = F(x) - F(x-1) = K(x)$. It follows $F(x+n) - F(x) = nK(x)$, whence $\frac{F(x+n)}{x+n} - \frac{F(x)}{n} = K(x)$.

From (a) the limit $\lim\limits_{t\to\infty} \frac{F(t)}{t}$ exists; denoting it by ℓ, we get for fixed x,

$$\lim_{n\to\infty} \frac{F(x+n)}{n} = \lim_{n\to\infty} \frac{x+n}{n} \frac{F(x+n)}{x+n} = 1 \cdot \ell = \ell.$$

Since $\lim\limits_{n\to\infty} \frac{F(x)}{n} = 0$ we obtain, for arbitrary $x \neq 0$, $K(x) = \ell$ (this proves also that $\ell \in \mathbf{R}$). Therefore $K(x)$ is a constant for $x \neq 0$ (and from continuity, for $x \in \mathbf{R}$), and f has period 1.

Suppose that f is not a constant and let $m = \min\limits_{x\in[0,1]} f(x)$. Then $F(1) - F(0) = \int_0^1 f(t)\,\mathrm{d}t > m$, whence $\ell = \lim\limits_{x\to\infty} \frac{F(x)}{x} = K(1) = F(1) - F(0) > m$. From $f(x) \geqslant \frac{F(x)}{x}$ if $x > 0$ it follows that there exists $\delta > 0$ such that $x > \delta \Rightarrow f(x) \geqslant \frac{F(x)}{x} > m$, which contradicts the periodicity of f.

PROBLEM 4. For a finite commutative group $(G, +)$ denote by $n(G)$ its cardinal and by $i(G)$ the number of algebraic operations $(G, *)$ such that $(G, +, *)$ is a ring (with unity). Prove that:

(a) $i(\mathbf{Z}_{12}) = 4$;

(b) $i(A \times B) \geqslant i(A)i(B)$, for every finite commutative groups A and B;

(c) there exist two sequences of finite commutative groups $(G_k)_{k\geqslant 1}$, $(H_k)_{k\geqslant 1}$ such that

$$\lim_{k\to\infty} \frac{n(G_k)}{i(G_k)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{n(H_k)}{i(H_k)} = \infty.$$

Solution. (a) If $(\mathbf{Z}_{12}, +, *)$ is a ring then

$$\hat{x} * \hat{y} = (\underbrace{\hat{1} + \cdots + \hat{1}}_{x \text{ times}}) * (\underbrace{\hat{1} + \cdots + \hat{1}}_{y \text{ times}}) = xy(\hat{1} * \hat{1})$$

for every $x, y \in \overline{0, n-1}$, so "$*$" is determined if we define $\hat{1} * \hat{1}$. If $\hat{u}$ is the multiplicative identity of the ring $(\mathbf{Z}_{12}, +, *)$ and $\hat{1}*\hat{1} = a\hat{1}$, $a \in \overline{0, 11}$, then $\hat{u}*\hat{1} = \hat{1}$ implies $ua\hat{1} = \hat{1}$, whence $ua \equiv 1 (\mathrm{mod}\, 12)$. This shows that a can be $1, 5, 7, 11$, so there are 4 posibilities to define $\hat{1} * \hat{1}$ (and they all work!) and $i(\mathbf{Z}_{12}) = 4$.

(b) If "$*$" and "$\triangle$" are such that $(A, +, *)$ and $(B, +, \triangle)$ are rings then $(A \times B, +, \circ)$, where $(a_1, b_1) \circ (a_2, b_2) = (a_1 * a_2, b_1 \triangle b_2)$ is a ring. Also, if "$*_1$" and "$*_2$" are different algebraic operations on A then the corresponding "$\circ_1$" and "$\circ_2$" are different. This shows that $i(A \times B) \geqslant i(A) \cdot i(B)$.

(c) Consider Klein group $(K, +)$, where $K = \{0, a, b, c\}$. Taking a as a multiplicative identity we can obtain at least two different structures of ring:

– one with $b^2 = b$, $c^2 = c$, $bc = cb = 0$ (corresponding to the ring $(\mathbf{Z}_2 \times \mathbf{Z}_2)$, with $a = (\hat{1}, \hat{1})$, $b = (\hat{1}, \hat{0})$, $c = (\hat{0}, \hat{1})$ and $0 = (\hat{0}, \hat{0})$);

– one with $b^2 = c$, $c^2 = b$, $bc = cb = a$ (corresponding to the field with 4 elements).

In the same way, taking b or c as multiplicative identity we can obtain other four different structures of ring, therefore $i(K) \geqslant 6$. Using (b) we get, for $G_p = (K, +)^p$, $i(G_p) \geqslant 6^p$, therefore

$$\frac{n(G_p)}{i(G_p)} \leqslant \frac{4^p}{6^p} \quad \text{and} \quad \lim_{p\to\infty} \frac{n(G_p)}{i(G_p)} = 0.$$

For the last part we notice that the same reasoning as in (a) shows that $i(\mathbf{Z}_n) = \varphi(n)$.

For n_k equal to the product of the first k prime numbers, $n_k = 2 \cdot 3 \cdot 5 \cdots p_k$ and $H_k = \mathbf{Z}_{n_k}$ we get $i(H_k) = (2-1)(3-1)\cdots(p_k - 1)$, whence

$$\frac{n(H_k)}{i(H_k)} = \prod_{i=1}^{k} \frac{p_i}{p_i - 1} := a_k.$$

The sequence $(a_k)_{k\geqslant 1}$ is clearly increasing so, if we prove that it is unbounded, we get $\lim\limits_{k\to\infty} a_k = \infty$. Indeed, if M is an arbitrary real then there exists an integer n such that $1 + \frac{1}{2} + \cdots + \frac{1}{n} > M$. Let $p_1, \ldots, p_{k_n}$ be the primes which appear in the decomposition of $2, 3, 4, \ldots, n$ and l be their largest exponent in these decompositions. Then

$$a_{k_n} = \prod_{i=1}^{k_n} \frac{1}{1 - \frac{1}{p_i}} > \prod_{i=1}^{k_n} \left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^l}\right) = \sum_{\alpha_1, \ldots, \alpha_{k_n} = 0}^{l} \frac{1}{p_1^{\alpha_1} \cdots p_{k_n}^{\alpha_{k_n}}}.$$

Among the denominators appear all the positive integers from 1 to n, therefore $a_{k_n} > 1 + \frac{1}{2} + \cdots + \frac{1}{n} > M$.

Part III. SELECTION EXAMINATIONS FOR THE INTERNATIONAL MATHEMATICAL OLYMPIAD BALKAN MATHEMATICAL OLYMPIAD AND JUNIOR BALKAN MATHEMATICAL OLYMPIAD

III.1. PROPOSED PROBLEMS

First selection examination for the 44th IMO and the 20th BMO

Sibiu, April 24, 2003

PROBLEM 1. Let $(a_n)_{n\geqslant 1}$ be the sequence defined by $a_1 = \frac{1}{2}$ and $a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}$ for $n \geqslant 1$. Prove that for any integer n, $n \geqslant 1$, we have

$$\sum_{k=1}^{n} a_k < 1.$$

Titu Andreescu

PROBLEM 2. A triangle ABC has $\angle A = 60°$. Suppose that P is a point with $PA = 1$, $PB = 2$, $PC = 3$. Find the maximal value of the area of triangle ABC.

* * *

PROBLEM 3. Let n, k be positive integers such that $n^k > (k+1)!$ and consider the set

$$M = \{(x_1, \ldots, x_n) \mid x_i \in \{1, 2, \ldots, n\},\, i = 1, \ldots, k\}.$$

Suppose A is a subset of M with $(k+1)!+1$ elements. Prove that there are $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ in A such that $(k+1)!$ divides $(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)\cdots(\beta_n - \alpha_n)$.

Vasile Zidaru

Second selection examination for the 44th IMO and the 20th BMO

Sibiu, April 25, 2003

PROBLEM 4. Consider the sequence $(a_n)_{n\geqslant 1}$ defined by $a_n = \lfloor n\sqrt{2003} \rfloor$ for $n \geqslant 1$. Prove that for any integers m and p, the sequence contains m elements in geometrical progresion of ratio greater than p.

Radu Gologan

PROBLEM 5. Let f be a irreducible polynomial, in $\mathbf{Z}[X]$ having highest degree coefficient 1 and such that $|f(0)|$ is not the square of an integer.

Prove that the polynomial g, given by $g(X) = f(X^2)$, is also irreducible.

Mihai Piticari

PROBLEM 6. In a mathematical competition $2n$ students take part. Each of the students submit a problem and all $2n$ problems collected in this way are given, one to one to the participants. The competition is considered "fair" if there are n competitors receaving the problems proposed by the other n competitors.

Prove that the number of ways in which the problems can be distributed in a "fair" competition is a perfect square.

* * *

Third selection examination for the 44th IMO

Piteşti, May 24, 2003

PROBLEM 7. Find all integers a, b, m, n, with $m > n > 1$, for which the polynomial $f(X) = X^n + aX + b$ divides the polynomial $g(X) = X^m + aX + b$.

Laurenţiu Panaitopol

PROBLEM 8. Two circles ω_1 and ω_2 with radii r_1 and r_2, $r_2 > r_1$, are externally tangent. The line t_1 is tangent to the circles ω_1 and ω_2 at points A and D respectively. The parallel line t_2 to the line t_1 is tangent to the circle ω_1 and intersects the circle ω_2 at points E and F. The line t_3 passing through D intersects the line t_2 and the circle ω_2 in B and C respectively, both different of E and F respectively. Prove that the circumcircle of the triangle ABC is tangent to the line t_1.

Dinu Şerbănescu

PROBLEM 9. Let $n \geqslant 3$ be a positive integer. Inside a $n \times n$ array there are placed n^2 positive numbers with sum n^3. Prove that we can find a square 2×2 of 4 elements of the array, having the sides parallel with the sides of the array, and for which the sum of the elements in the square is greater than $3n$.

Radu Gologan

Fourth selection examination for the 44th IMO

Piteşti, April 25, 2003

PROBLEM 10. Let $\mathcal{P}$ the set of all the primes and let M be a subset of $\mathcal{P}$, having at least three elements, and such that for any proper subset A of M all of the prime factors of the number

$$-1 + \prod_{p \in A} p$$

are found in M. Prove that $M = \mathcal{P}$.

Valentin Vornicu

PROBLEM 11. In a square of side 6 the points A, B, C, D are given such that the distance between any two of the four points is at least 5. Prove that A, B, C, D form a convex quadrilateral and its area is greater than 21.

Laurenţiu Panaitopol

PROBLEM 12. A word consists of n letters from the alphabet $\{a, b, c, d\}$. One says that a word is complicated if it has two consecutive identical groups of letters (i.e. *caab* or *cababdc* are complicated words, but *abcab* is not a complicated word). A word that is not complicated is called a *simple word.*

Prove that the number of simple words with n letters is greater than 2^n.

* * *

Fifth selection examination for the 44th IMO

Bucharest, June 19, 2003

PROBLEM 13. A country's parliament has n members. Each belongs to exactly one party and exactly one commission.

Find the minimum value of n for which in any numerical distribution of the parties and the commissions, there is a numerotation with numbers $1, 2, \ldots, 10$ of the parties and of the commissions such that at least 11 members belong to a party and a commission with the same number each.

Marian Andronache, Radu Gologan

PROBLEM 14. Consider a rhombus $ABCD$ of side 1. On sides BC and CD there are points M, N respectively, such that $CM + MN + NC = 2$ and $\angle MAN = \frac{1}{2}\angle BAD$.

Find the angles of the rhombus.

Cristinel Mortici

PROBLEM 15. In a Cartesian plane XOY a point $A(x, y)$ is called a *lattice* point if x and y are integers. A lattice point B is called *invisible* if on the open segment (OA) there is a lattice point.

Prove that for any positive integer n there is a square of side n having all points in the interior or on the boundary invisible.

France qualification tests, 2003

Sixth selection examination for the 44th IMO
Bucharest, June 20, 2003

PROBLEM 16. Let $ABCDEF$ be a convex hexagon and denote by A', B', C', D', E', F' the middle points of the sides AB, BC, CD, DE, EF and FA respectively. Given are the areas of the triangles ABC', BCD', CDE', DEF', EFA' and FAB'. Find the area of the hexagon.

Kvant

PROBLEM 17. A permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is called *straight* if and only if for each integer k, $1 \leqslant k \leqslant n-1$ the following inequality is fulfilled

$$|\sigma(k) - \sigma(k+1)| \leqslant 2.$$

Find the smallest positive integer n for which there exist at least 2003 straight permutations.

Valentin Vornicu

PROBLEM 18. For every positive integer n we denote by $d(n)$ the sum of its digits in the decimal representation. Prove that for each positive integer k there exists a positive integer m such that the equation $x + d(x) = m$ has exactly k solutions in the set of positive integers.

Mihai Manea

First team selection test for the Junior Balkan Mathematical Olympiad
Sibiu, May 23, 2003

PROBLEM 1. Consider a rhombus $ABCD$ with center O. A point P is given inside the rhombus, but not situated on the diagonals. Let M, N, Q, R be the projection of P on the sides $(AB), (BC), (CD), (DA)$, respectively. The perpendicular bisectors of the segments MN and RQ meet at S and the perpendicular bisectors of the segments NQ and MR meet at T.

Prove that P, S, T and O are the vertices of a rectangle.

Mircea Fianu

PROBLEM 2. Consider the prime numbers $n_1 < n_2 < \cdots < n_{31}$. Prove that if 30 divides $n_1^4 + n_2^4 + \cdots + n_{31}^4$, then among these numbers one can find three consecutive primes.

Vasile Berghea

PROBLEM 3. Let n be a positive integer. Prove that there are no positive integers x and y such as

$$\sqrt{n} + \sqrt{n+1} < \sqrt{x} + \sqrt{y} < \sqrt{4n+2}.$$

Dinu Şerbănescu

PROBLEM 4. Show that one can color all the points of a plane using only two colors such that no line segment has all points of the same color.

Valentin Vornicu

Second team selection test for the Junior Balkan Mathematical Olympiad
Piteşti, May 24, 2003

PROBLEM 5. Let a, b, c be positive real numbers with $abc = 1$. Prove that

$$1 + \frac{3}{a+b+c} \geqslant \frac{6}{ab+bc+ca}.$$

Mircea Lascu and Vasile Cârtoaje

PROBLEM 6. Two circles $\mathcal{C}_1(O_1)$ and $\mathcal{C}_2(O_2)$ with distinct radii meet at points A and B. The tangent from A to $\mathcal{C}_1$ intersects the tangent from B to $\mathcal{C}_2$ at point M.

Show that both circles are seen from M under the same angle.

Dinu Şerbănescu

PROBLEM 7. Five real numbers of absolute values not greater than 1 and having the sum equal to 1 are written on the circumference of a circle.

Prove that one can choose three consecutively disposed numbers a, b, c, such that all the sums $a+b, b+c$ and $a+b+c$ are nonnegative.

Dinu Şerbănescu

PROBLEM 8. Let E be the midpoint of the side CD of a square $ABCD$. Consider the point M inside the square such that

$$\angle MAB = \angle MBC = \angle BME = x.$$

Find the angle x.

Laurenţiu Panaitopol

Third team selection test for the Junior Balkan Mathematical Olympiad

Piteşti, May 25, 2003

PROBLEM 9. Suppose $ABCD$ and $AEFG$ are rectangles such that the points B, E, D, G are collinear (in this order). Let the lines BC and GF intersect at point T and let the lines DC and EF intersect at point H. Prove that points A, H and T are collinear.

Mircea Fianu

PROBLEM 10. Let a be a positive integer such that the number a^n has an odd number of digits in the decimal representation, for all $n > 0$. Prove that the number a is an even power of 10.

Vasile Zidaru

PROBLEM 11. A set of 2003 positive integers is given. Show that one can find two elements such that their sum is not a divisor of the sum of the other elements.

Valentin Vornicu

PROBLEM 12. Two unit squares with parallel sides overlap by a rectangle of area 1/8. Find the extreme values of the distance between the centers of the squares.

Radu Gologan

III.2. SELECTION EXAMINATIONS – SOLUTIONS

First selection examination for the 44th IMO and the 20th BMO

PROBLEM 1. Let $(a_n)_{n\geqslant 1}$ be the sequence defined by $a_1 = \frac{1}{2}$ and $a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}$ for $n \geqslant 1$. Prove that for any integer n, $n \geqslant 1$, we have

$$\sum_{k=1}^{n} a_k < 1.$$

Solution. If $b_n = \frac{1}{a_n}$ then $b_1 = 1$ and $b_{n+1} = b_n^2 - b_n + 1$ for any $n \geqslant 1$. We deduce $b_{n+1} - 1 = b_n(b_n - 1)$. By multiplication

$$b_{n+1} = b_1 \cdots b_n + 1$$

or

$$\frac{1}{b_n + 1} + \frac{1}{b_1 \cdots b_n b_{n+1}} = \frac{1}{b_1 \cdots b_n}.$$

Summing up one obtain

$$\frac{1}{b_1} + \cdots + \frac{1}{b_n} + \frac{1}{b_1 \cdots b_n} = 1$$

whence the conclusion holds.

PROBLEM 2. A triangle ABC has $\angle A = 60°$. Suppose that P is a point with $PA = 1$, $PB = 2$, $PC = 3$. Find the maximal value of the area of triangle ABC.

Solution. Consider the parallelograms $ABCD$ and $APCE$. Then $BEDP$ is also a parallelogram. We have:

$$\sigma(ABC) = \frac{1}{2} AB \cdot AC \cdot \sin 60° = \frac{\sqrt{3}}{4} AB \cdot AC$$

and by Ptolemey's relation

$$\frac{\sqrt{3}}{4}AB\cdot AC=\frac{\sqrt{3}}{4}CD\cdot PE\leqslant\frac{\sqrt{3}}{4}(DE\cdot PC+CE\cdot PD)=\frac{\sqrt{3}}{4}(6+PD).$$

On the other side

$$PA^2+PD^2-PB^2-PC^2=2PO^2+\frac{1}{2}AD^2-2PO^2-\frac{1}{2}BC^2$$
$$=2AB\cdot AC\cdot\sin 60^\circ=AB\cdot AC$$

$$PD^2=12+AB\cdot AC=12+CD\cdot PE\leqslant 12+PC\cdot ED+PD\cdot CE.$$

Thus $PD^2\leqslant 12+6+PD$ and $PD\leqslant\frac{1+\sqrt{73}}{2}$.

The maximum is this $\sigma_{\max}(ABC)=\frac{\sqrt{3}}{8}(13+\sqrt{73})$ and we have equality when $PCED$ is cyclic, that is $\angle PCA=\angle PBA$

PROBLEM 3. Let n,k be positive integers such that $n^k>(k+1)!$ and consider the set

$$M=\{(x_1,\ldots,x_n)\mid x_i\in\{1,2,\ldots,n\},\ i=1,\ldots,k\}.$$

Suppose A is a subset of M with $(k+1)!+1$ elements. Prove that there are $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$ in A such that $(k+1)!$ divides $(\beta_1-\alpha_1)(\beta_2-\alpha_2)\cdots(\beta_n-\alpha_n)$.

Solution. Consider that function $f:A\to\mathbf{Z}_2\times\cdots\times\mathbf{Z}_{k+1}=B$ (here $\mathbf{Z}_{k+1}=\{\hat{0},\hat{1},\ldots,\hat{k}\}$ defined by

$$f(\gamma_1,\ldots,\gamma_k)=(\hat{\gamma_1},\hat{\gamma_2},\ldots,\hat{\gamma_k}).$$

As $\operatorname{card}A=(k+1)!+1$ and $\operatorname{card}B=(k+1)!$, the function is not injective. The fact that $f(\alpha_1,\ldots,\alpha_k)=f(\beta_1,\ldots,\beta_k)$ gives the result.

Second selection examination for the 44th IMO and the 20th BMO

PROBLEM 4. Consider the sequence $(a_n)_{n\geqslant 1}$ defined by $a_n=\lfloor n\sqrt{2003}\rfloor$ for $n\geqslant 1$. Prove that for any integers m and p, the sequence contains m elements in geometrical progresion of ratio greater than p.

Solution. By any irrational x, by Kronecker's theorem we can find a positive integer n such that $nx=N_n+\alpha_n$ where N_n is a positive integer and $0<\alpha_n<\frac{1}{2^{m+1}}$. Then $2^knx=2^kN_n+2^k\alpha_n$ and $\lfloor 2^knx\rfloor=2^k\lfloor nx\rfloor$ for $k=0,\ldots,m$. Thus

$(\lfloor 2^knx\rfloor)_{k=0,\ldots,m}$ is a geometrical progresion of ratio 2. For a longer ratio it is sufficient to use the result for a subprogresion of ratio say $2^r>p$.

PROBLEM 5. Let f be a irreducible polynomial in $\mathbf{Z}[X]$ having highest degree coefficient 1 and such that $|f(0)|$ is not the square of an integer.

Prove that the polynomial g, given by $g(X)=f(X^2)$, is also irreducible.

Solution. Suppose that we can find non-constant polynomials in $\mathbf{Z}[X]$ with $\deg p,\deg q<2n$, where $n=\deg f$, such that $g=p\cdot q$.

If $\alpha\in\mathbf{C}$ such that $f(\alpha)=0$ we have $p(\sqrt{\alpha})=0$. We thus obtain integer polynomials t,u such that

$$t(\alpha)+\sqrt{\alpha}u(\alpha)=0$$

with $\deg t,\deg u<2n\leqslant\frac{1}{2}\deg p<n$. As $u\neq 0$ and u,f are relatively prime, there are $s,r\in\mathbf{Q}[X]$ such that

$$su+rf=1.$$

Thus $s(\alpha)u(\alpha)=1$ which implies $\sqrt{\alpha}=-t(\alpha)s(\alpha)$ or $\alpha=t^2(\alpha)s^2(\alpha)$. If m is the polynomial t^2s^2-X then $m(\alpha)=0$ which implies $f\mid m$.

If $\alpha_1,\ldots,\alpha_n\in\mathbf{C}$ are the zeros of f then $m(\alpha_i)=0$, $i=1,\ldots,n$, and $\alpha_i=t^2(\alpha_i)s^2(\alpha_i)$. We deduce

$$\alpha_1\cdots\alpha_n=(ts)^2(\alpha_1)(ts)^2(\alpha_2)\cdots(ts)^2(\alpha_n)$$

the square of a rational number. As $f(0)$ is an integer and $f(0)=(-1)^n\alpha_1\cdots\alpha_n$ we get a contradiction.

PROBLEM 6. In a mathematical competition $2n$ students take part. Each of the students submit a problem and all $2n$ problems collected in this way are given, one to one to the participants. The competition is considered "fair" if there are n competitors receaving the problems proposed by the other n competitors.

Prove that the number of ways in which the problems can be distributed in a "fair" competition is a perfect square.

Solution. Any distribution of the problems corresponds to a permutation of the set $\{1,2,\ldots,2n\}=A_n$. Any such permutation is associated with a fair contest if the set A_n can be written as the union of disjoint sets M_1 and M_2 each of n elements such that elements in M_1 have images in M_2. This corresponds to the fact that the permutation has only even cycles.

Let a_n be the number of such permutations. The number of permutations for which 1 is an element of a cycle of order $2k$ is $\binom{2n-1}{2k-1}\cdot(2k-1)!a_{n-k}$. In fact, we can choose the elements of the cycle in $\binom{2n-1}{2k-1}$, we can order them in $(2k-1)!$

ways and the rest of the permutation can be completed in a_{n-k} ways. It is easy to see that $a_0 = 1$, $a_1 = 1$ and then

$$a_n = \sum_{k=1}^{n} \binom{2n-1}{2k-1}(2k-1)!a_{n-k} = (2n-1)a_{n-1} + \sum_{k=2}^{n} \binom{2n-1}{2k-1}(2k-1)!a_{n-k}$$
$$= (2n-1)a_{n-1} + (2n-1)(2n-2)\sum_{k=1}^{n-1}\binom{2n-3}{2k-1}(2k-1)!a_{n-k-1}$$
$$= (2n-1)a_{n-1} + (2n-1)(2n-2)a_{n-1} = (2n-1)^2 a_{n-1}$$

which complets the result by induction.

Third selection examination for the 44th IMO

PROBLEM 7. Find all integers a, b, m, n, with $m > n > 1$, for which the polynomial $f(X) = X^n + aX + b$ divides the polynomial $g(X) = X^m + aX + b$.

Solution. It is obvious that the solutions $(0, 0, m, n)$ work. If $a = 0$, $b \neq 0$ then $X^n + b \mid X^m + b$, thus all the roots of f are also the roots of g, and thus $|b| = 1$. If $b = -1$ then we must have $X^n - 1 \mid X^m - 1$ and using Euclid's algorithm we obtain that $\gcd(X^n - 1, X^m - 1) = X^{\gcd(m,n)} - 1$, thus $n \mid m$. It follows that another set of solutions is $(0, -1, kn, n)$, with k being any positive integer.

If $b = 1$ then the roots of f have the form

$$z_k = \cos\left(\frac{2k\pi}{n} + \pi\right) + i\sin\left(\frac{2k\pi}{n} + \pi\right), \quad \forall k = \overline{0, n-1}$$

and they must also be the roots of g thus we have

$$(1) \qquad \left(\frac{2k\pi}{n} + \pi\right)m = (2k' + 1)\pi \Rightarrow \frac{2km}{n} + m = 2k' + 1 .$$

It follows from (1) that $n \mid 2km$ for each $k = \overline{0, n-1}$. In particularly, for $k = 1$, which means that $2m = \alpha n$. Therefore (1) becomes $k\alpha + m = 2k' + 1$. But if α is not even, then for even k and odd k be obtain different parities of $2k' + 1$ which is a contradiction; thus $\alpha = 2\alpha'$ and m is odd, which also leads to $m = \alpha' n$ implying that n and α' are also odd. Thus the third set of solutions are $(0, 1, (2k+1)n, n)$, for any positive integer k.

If $b = 0$ and $a \neq 0$ then we have $X^{n-1} + a \mid X^{m-1} + a$, and using the same arguments as above we find two new sets of solutions: $(-1, 0, kn - k + 1, n)$ and $(1, 0, (2k+1)(n-1) + 1, n)$.

Finally, suppose that $a \neq 0 \neq b$. Then $f(x) \mid g(x)$ implies that $f(x) \mid g(x) - f(x)$, thus $X^n + aX + b \mid X^m - X^n$, and furthermore $X^n + aX + b \mid X^{m-n} - 1$. The last relation implies that all of f's roots are unit roots, thus $|b| = 1 \Rightarrow b \in \{-1, 1\}$. Furthermore if z_i is a root of $X^{m-n} - 1$ and a root of f, $i = \overline{1, n}$, then $\overline{z_i} = \frac{1}{z_i}$ is also a root of f (because f has real coefficients). From Viète's relationships we obtain that

$$a = (-1)^{n-1}\prod_{i=1}^{n} z_i \left(\sum_{i=1}^{n} \frac{1}{z_i}\right) \Rightarrow a = (-1)^{2n-1} b \sum_{i=1}^{n} \overline{z_i}$$

But if $n > 2$ then $\overline{z_1} + \overline{z_2} + \cdots + \overline{z_n} = z_1 + \cdots + z_n = 0$ thus $a = 0$ and we have returned to one of the previous cases. Thus we must have $n = 2$, and in that case $a = (-b) \cdot (-a) \Rightarrow b = 1$. We also have $x^2 + ax + 1 = 0 \Leftrightarrow |a| = |ax| = |1 + x^2| \leqslant 1 + |x|^2 = 2$, thus $a \in \{-2, -1, 1, 2\}$. If $a = 2$ or $a = -2$ then $X^{m-n} - 1$ would have a double root — which is a contradiction, thus the only possibilities are $a = -1$ and $a = 1$. If $a = 1$, it is easy to see that $m - 2 = 3k$ and thus the solutions are $(1, 1, 3k + 2, 2)$, for all positive integers k and if $a = -1$ we have $m - 2 = 6k$, which leads to the solutions $(-1, 1, 6k + 2, 2)$ for all positive integers k.

PROBLEM 8. Two circles ω_1 and ω_2, with radii r_1 and r_2, $r_2 > r_1$, are externally tangent. The line t_1 is tangent to the circles ω_1 and ω_2 at points A and D respectively. The parallel line t_2 to the line t_1 is tangent to the circle ω_1 and intersects the circle ω_2 at points E and F. The line t_3 passing through D intersects the line t_2 and the circle ω_2 in B and C respectively, both different of E and F respectively. Prove that the circumcircle of the triangle ABC is tangent to the line t_1.

Solution. It is enough to prove that $AD^2 = DB \cdot DC$. It is easy to compute the length of the common tangent: $AD^2 = 4r_1r_2$.

Let now $DM \perp EF$, $M \in EF$ and $N = DM \cap \omega_2$. Then the right-angled triangles DBM and DNC are similar, hence $DB \cdot DC = DM \cdot DN = 2r_1 \cdot 2r_2 = 4r_1r_2$.

PROBLEM 9. Let $n \geqslant 3$ be a positive integer. Inside a $n \times n$ array there are placed n^2 positive numbers with sum n^3. Prove that we can find a square 2×2 of 4 elements of the array, having the sides parallel with the sides of the array, and for which the sum of the elements in the square is greater than $3n$.

Solution. Let $a_1, a_2, \ldots, a_{n^2}$ be the array's elements and $S_1, S_2, \ldots, S_k$ be all the sums of the elements of the possible squares. If $S_i \leqslant 3n$ for all i, then

$$S_1 + S_2 + \cdots + S_k \leqslant 3n \cdot \frac{n(n+1)(2n+1)}{6}.$$

On the other hand, by counting the number of times each element of the array appears in the above sum, one obtains:

$$\begin{aligned}&S_1+S_2+\cdots+S_k\\&\quad\geqslant(n-1)(a_1+\cdots+a_{4n-4})+(n+1)(a_{4n-5}+\cdots+a_{8n-18})+\cdots\\&\quad\geqslant(n-1)\sum_{i=1}^{n^2}a_i\end{aligned}$$

$$\Rightarrow 6(n-1)(n+1)^3\leqslant 3n^2(n+1)(2n+1)\Leftrightarrow 2n^3+2n^2-2n-2\leqslant 2n^3+n^2$$

which is false for $n\geqslant 3$, and thus the problem is solved.

Fourth selection examination for the 44th IMO

PROBLEM 10. Let $\mathcal{P}$ the set of all the primes and let M be a subset of $\mathcal{P}$, having at least three elements, and such that for any proper subset A of M all of the prime factors of the number

$$-1+\prod_{p\in A}p$$

are found in M. Prove that $M=\mathcal{P}$.

Solution. If $2\notin M$, then take $A=\{p\}$, with $p\in M$. Because $p-1$ is an even number, it follows that $2\in M$, contradiction, thus $2\in M$.

First let us suppose that M is finite. Then let M be $\{2,p_2,\ldots,p_k\}$, $k\geqslant 3$. Let A be $\{2,p_3,\ldots,p_k\}$, and denote by P the product of all the elements of M. Then we have:

$$(1)\qquad 2p_3\cdots p_k-1=\frac{P}{p_2}-1=p_2^{\alpha}\Rightarrow P=p_2^{\alpha+1}+p_2$$

and if we consider $A=\{p_3,\ldots,p_k\}$ then we obtain:

$$(2)\qquad p_3\cdots p_k-1=\frac{P}{2p_2}-1=2^{\beta}p_2^{\gamma}\Rightarrow P=2p_2(2^{\beta}p_2^{\gamma}+1).$$

From (1) and (2) it follows that

$$p_2^{\alpha+1}+p_2=2p_2(2^{\beta}p_2^{\gamma}+1)\Rightarrow p_2\alpha+1=2^{\beta+1}p_2^{\gamma}+2\Rightarrow 1\equiv 2(\operatorname{mod}p_2)$$

contradiction, thus indeed M has an infinite number of elements.

Suppose now that there is a prime q such that $q\notin M$. Let

$$M=\{2,p_2,p_3,\ldots,p_k,\ldots\}.$$

From the Pigeonhole principle it follows that from the numbers $2-1, 2\cdot p_2-1,\ldots,2p_2\cdots p_{q+1}-1$ at least two of them have the same residue modulo q, let them be $2\cdots p_i-1\equiv 2\cdots p_j-1$, $1\leqslant i<j\leqslant q+1$. But then we have

$$2\cdots p_i(p_{i+1}\cdots p_j-1)\equiv 0(\operatorname{mod}q)\Rightarrow p_{i+1}\cdots p_j-1\equiv 0(\operatorname{mod}q)\Rightarrow q\in M$$

which is a contradiction. Therefore the supposition was false and M is the set of all primes.

PROBLEM 11. In a square of side 6 the points A,B,C,D are given such that the distance between any two of the four points is at least 5. Prove that A,B,C,D form a convex quadrilateral and its area is greater than 21.

Solution. First of all we observe that no angle formed with 3 of the 4 points can be greater of equal with $120°$, because otherwise if we suppose that WLOG $\angle ABC\geqslant 120°$, then from $AB\geqslant 5$ and $BC\geqslant 5$ we deduce $AC\geqslant 5\sqrt{3}>6\sqrt{2}$ contradiction.

Therefore if the quadrilateral $ABCD$ is not convex, then one of the 4 points lies inside the triangle formed by the other 3. Suppose WLOG that $D\in\operatorname{int}[ABC]$. But then one of the angles $\angle ADB$, $\angle BDC$ and $\angle CDA$ would be, by the Pigeonhole principle, greater or equal than $120°$, contradiction. Thus $ABCD$ is a convex quadrilateral.

Now because each angle of the triangle ABC is smaller than $120°$ and there is at least one angle, say $\angle ABC$, which is greater than $60°$ it follows that

$$\sin\angle ABC\geqslant\frac{\sqrt{3}}{2}$$

$$\Rightarrow\sigma[ABC]=\frac{1}{2}AB\cdot BC\cdot\sin\angle ABC\geqslant\frac{\sqrt{3}}{4}\cdot 25>\frac{21}{2}\Leftrightarrow 625>12\cdot 49=588.$$

Analogously one can prove that $\sigma[ACD]\geqslant\frac{21}{2}$, and thus $\sigma[ABCD]>21$.

PROBLEM 12. A word consists of n letters from the alphabet $\{a,b,c,d\}$. One says that a word is complicated if it has two consecutive identical groups of letters (i.e. *caab* or *cababdc* are complicated words, but *abcab* is not a complicated word). A word that is not complicated is called a simple word.

Prove that the number of simple words with n letters is greater than 2^n.

Solution. Let us denote by $S(n)$ the set of simple words with n letters and by s_n the number of elements in $S(n)$. If we put a letter at the end of each of the simple words from $S(n)$ we obtain a set $T(n+1)$ of t_{n+1} words of length $n+1$, their number being $t_{n+1} = 4s_n$. Obviously, $S(n+1) \subset T(n+1)$, $S(n+1) \neq T(n+1)$.

Let $T_1(n+1)$ be the set of those words from $T(n+1)$ which have the last two letters the same, $T_k(n+1)$ the set of $T(n+1)$ which end in two consecutive identical groups of k letters, for each $k \in \{1, 2, \ldots, m\}$, where $m = \left\lfloor \frac{n+1}{2} \right\rfloor$. Obviously

$$f(n+1) \geqslant t_{n+1} - |T_1(n+1)| - |T_2(n+1)| - \cdots - |T_m(n+1)|$$

We have $t_{n+1} = 4f(n)$, $|T_1(n+1)| = f(n)$, and furthermore $|T_k(n+1)| \leqslant f(n+1-k)$, because of the fact that the mapping of $S(n+1-k)$ into $T_k(n+1)$ given by adding to a simple word of $n+1-k$ letters its own last k letters is obviously surjective.

We have $f(1) = 4$, $f(2) = 12 > 4$. By induction we want to prove $f(k+1) > 2f(k)$. We have

$$f(n+1) \geqslant 4f(n) - f(n) - \frac{1}{2}f(n) - \frac{1}{4}f(n) - \cdots > 2f(n)$$

from which the conclusion follows.

Fifth selection examination for the 44th IMO

PROBLEM 13. A country's parliament has n members. Each belongs to exactly one party and exactly one commission.

Find the minimum value of n for which in any numerical distribution of the parties and the commissions, there is a numerotation with numbers $1, 2, \ldots, 10$ of the parties and of the commissions such that at least 11 members belong to a party and a commission with the same number each.

Solution. It is clear that if $n = 100$ we can consider a configuration where each party has exactly ten members, each belonging to a different commission. Suppose that parties are numbered $1, \ldots, 10$. In any numbering of the commission there will be exactly 10 parliament members having the same party-commission number. It is then clear that any smaller n does not satisfy the condition. Thus $n > 100$.

Let us prove that $n = 101$ is a "good" number.

Consider $A_1, \ldots, A_{10}$ the party partition and $B_1, \ldots, B_{10}$ the commission partition of the parliament. It is clear that we can consider that $A_1, \ldots, A_{10}$ is also a fixed numbering of parties. By way of contradiction suppose that for any numbering of the commissions there are less than 11 members with same numbers. As a numbering of the B_i corresponds to a permutation σ of $\{1, \ldots, 10\}$, we should have:

$$\sum_{i=1}^{10} |A_i \cap B_{\sigma(i)}| \leqslant 10.$$

Summing over σ we get

$$\sum_{\sigma \in S_{10}} \sum_{i=1}^{10} |A_i \cap B_{\sigma(i)}| \leqslant 10 \cdot 10!$$

It is easy to see that the left sum is $9! \cdot 101$. This is because each element of the set of parlament members appears in $9!$ sets of the form $A_i \cap B_{\sigma(i)}$.

We thus get $9! \cdot 101 \leqslant 10 \cdot 10!$ or $101 \leqslant 100$, a contradiction.

PROBLEM 14. Consider a rhombus $ABCD$ of side 1. On sides BC and CD there are points M, N respectively, such that $CM + MN + NC = 2$ and $\angle MAN = \frac{1}{2}\angle BAD$.

Find the angles of the rhombus.

Solution. Let $ADC'D'$ the rhombus obtained by rotating the initial one around A with angle BAD.

By hypothesis, triangles AMN and $AM'N$ are equal, thus $M'N = MN = x + y$, because the perimeter of MNC is 2.

In conclusion, $M'N = x + y = M'D + DN$, thus M', D, N are collinear, implying that $ABCD$ is a square.

Reciprocaly, if $ABCD$ is a square, by the same construction, if $\angle MAN = \frac{1}{2}\angle BAD$ then the perimeter of MCN is 2.

PROBLEM 15. In a Cartesian plane XOY a point $A(x, y)$ is called a *lattice* point if x and y are integers. A lattice point B is called *invisible* if on the open segment (OA) there is a lattice point.

Prove that for any positive integer n there is a square of side n having all points in the interior or on the boundary invisible.

Solution. Consider an array $A = (p_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$ consisting of disting primes p_{ij} and σ the cyclic permutation $(1,\dots,n) \mapsto (n,1,\dots,n-1)$. Define the numbers:

$$a_i = \prod_{k=1}^{n} p_{ik}, \quad i = 1,\dots,n$$
$$b_i = \prod_{k=1}^{n} p_{k\sigma^{i-1}(k)}, \quad i = 1,\dots,n.$$

It is clear that $(a_i, a_j) = 1$, $(b_i, b_j) = 1$ for $i \neq j$ and that $(a_i, b_j) \neq 1$ for any $i, j \in \{1,\dots,n\}$.

By the Remainder Chimise Lemma we can find x, y positive integers such that

$$x \equiv -(k-1)(\operatorname{mod} a_k) \quad \text{for } k = 1,\dots,n$$

and

$$y \equiv -(k-1)(\operatorname{mod} b_k) \quad \text{for } k = 1,\dots,n.$$

These mean that there are nonnegative integers $k_1,\dots,k_n$, $l_1,\dots,l_n$ such that

$$k_{i+1}a_{i+1} = k_i a_i + 1 \quad \text{and} \quad l_{i+1}b_{i+1} = l_i b_i + 1$$

for any $i = 1,\dots,n-1$.

The square $ABCD$ defined by $A(k_1a_1, l_1b_1)$, $B(k_na_n, l_1b_1)$, $C(k_na_n, l_nb_n)$ and $D(k_1a_1, l_nb_n)$ has the desired properties.

Sixth selection examination for the 44th IMO

PROBLEM 16. Let $ABCDEF$ be a convex hexagon and denote by A', B', C', D', E', F' the middle points of the sides AB, BC, CD, DE, EF and FA respectively. Given are the areas of the triangles ABC', BCD', CDE', DEF', EFA' and FAB'. Find the area of the hexagon.

Solution. Let us denote by $\sigma[ABC]$ the area of the triangle ABC and by S the area of the hexagon $ABCDEF$. We shall use the following lemma:

LEMMA. *In a quadrilateral $ABCD$ take M to be the middle point of the side CD. Then the area of the triangle ABM is the arithmetic mean value of the areas of the triangles ABC and ABD.*

Proof of the lemma is obtained easily by drawing the altitudes from C, M, D to the side AB, and using the fact that the altitude from M is the middle line in the right-angled trapezoid formed with the altitudes from C and D.

Now we split the area of the hexagon in 4 areas of triangles:

$$S = \sigma[ABD] + \sigma[ADE] + \sigma[BCD] + \sigma[AEF] \tag{1}$$
$$S = \sigma[BCE] + \sigma[BEF] + \sigma[CDE] + \sigma[FAB] \tag{2}$$
$$S = \sigma[DEA] + \sigma[ACD] + \sigma[ABC] + \sigma[DEF]. \tag{3}$$

Using the lemma and summing up all the relationships (1), (2) and (3) we obtain:

$$3S = 2\sigma[ABC'] + 2\sigma[BCD'] + 2\sigma[CDE'] + 2\sigma[DEF'] + 2\sigma[EFA'] + 2\sigma[FAB']$$

$$\Rightarrow S = \frac{2}{3}\left(\sigma[ABC'] + \sigma[BCD'] + \sigma[CDE'] + \sigma[DEF'] + \sigma[EFA'] + \sigma[FAB']\right).$$

PROBLEM 17. A permutation $\sigma : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ is called *straight* if and only if for each integer k, $1 \leqslant k \leqslant n-1$ the following inequality is fulfilled

$$|\sigma(k) - \sigma(k+1)| \leqslant 2.$$

Find the smallest positive integer n for which there exist at least 2003 straight permutations.

Solution. The main ideea is to look where n is positioned. In that idea let us denote by x_n the number of all the straight permutations and by a_n the number of straight permutations having n on the first or on the last position, i.e. $\sigma(1) = n$ or $\sigma(n) = n$. Also let us denote by b_n the difference $x_n - a_n$ and by a'_n the number of permutations having n on the first position, and by a''_n the number of permutations having n on the last position. From symmetry we have that $2a'_n = 2a''_n = a'_n + a''_n = a_n$, for all n-s. Therefore finding a recurrence relationship for $\{a_n\}_n$ is equivalent with finding one for $\{a'_n\}_n$.

One can simply compute: $a'_2 = 1$, $a'_3 = 2$, $a'_4 = 4$. Suppose that $n \geqslant 5$. We have two possibilities for the second position: if $\sigma(2) = n-1$ then we must complete the remaining positions with $3, 4, \dots, n$ thus the number of ways in which we can do that is a'_{n-1} (because the permutation $\sigma' : \{1,2,\dots,n-1\} \to \{1,2,\dots,n-1\}$, $\sigma'(k) = \sigma(k+1)$, for all k, $1 \leqslant k \leqslant n-1$, is also a straight permutation).

If on the second position we have $n-2$, $\sigma(2) = n-2$, then $n-1$ can only be in the last position of the permutation or on the third position, i.e. $\sigma(3) = n-1$ or $\sigma(n) = n-1$. If $\sigma(n) = n-1$, then we can only have $\sigma(n-1) = n-3$ thus $\sigma(3) = n-4$ and so on, thus there is only one permutation of this kind.

On the other hand, if $\sigma(3) = n-1$ then it follows that $\sigma(4) = n-3$ and now we can complete the permutation in a'_{n-3} ways (because the permutation $\sigma' : \{1,2,\ldots,n-3\} \to \{1,2,\ldots,n-3\}$, $\sigma'(k) = \sigma(k+3)$, for all k, $1 \leqslant k \leqslant n-3$, is also a straight permutation).

Summing all up we get the recurrence:

$$(1) \qquad a'_n = a'_{n-1} + 1 + a'_{n-3} \Rightarrow a_n = a_{n-1} + a_{n-3} + 2, \quad \forall n \geqslant 5.$$

The recurrence relationship for $\{b_n\}$ can be obtained by observing that for each straight permutation $\tau : \{1,2,\ldots,n+1\} \to \{1,2,\ldots,n+1\}$ for which $2 \leqslant \tau^{-1}(n+1) \leqslant n$ we can obtain a straight permutation $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ by removing $n+1$. Indeed $n+1$ is "surrounded" by n and $n-1$, so by removing it, n and $n-1$ become neighbours, and thus the newly formed permutation is indeed straight. Now, if $\tau^{-1}n \in \{1, n+1\}$ then the newly formed permutation σ was counted as one of the a_n-s, minus the two special cases in which n and $n-1$ are on the first and last positions, and also minus the permutations which begin or end with a sequence of the form $n, n-2, n-1, n-3, \ldots$ thus n and $n-1$ not being neighbours. If $\tau^{-1}(n) \notin \{1, n+1\}$ then certainly σ was counted with the b_n-s. Also, from any straight permutation of n elements, not having n and $n-1$ in the first and last position, thus n certainly being neighbour with $n-1$, we can make a straight $n+1$-element permutation by inserting $n+1$ between n and $n-1$.

Therefore we have obtained the following relationship:

$$\begin{aligned} &b_{n+1} = a_n - 2 - a_{n-3} + b_n = a_{n-1} + b_n, \\ &\Rightarrow b_n = a_{n-2} + a_{n-3} + \cdots + a_2 + b_3 \\ &\Rightarrow x_n = a_n + a_{n-2} + a_{n-3} + \cdots + a_2 + b_3, \quad \forall n \geqslant 4. \end{aligned}$$

It is obvious that $\{x_n\}_n$ is a "fast" increasing sequence, so it is easy to compute the first terms using the relationships obtained above, which will prove that the number that we are looking for is $n = 16$.

PROBLEM 18. For every positive integer n we denote by $d(n)$ the sum of its digits in the decimal representation. Prove that for each positive integer k there exists a positive integer m such that the equation $x + d(x) = m$ has exactly k solutions in the set of positive integers.

Solution. Let us denote by $f(x) = x + d(x)$, for each positive integer x. We shall prove by induction after k that we can find the numbers $x_1 < x_2 < \cdots < x_k$, such that x_k begins with the digit 1, and has at least two digits, and $f(x_1) = f(x_2) = \cdots = f(x_k)$.

For $k = 1$ just take $x_1 = 11$.

Suppose that the statement holds for k, and let $x_1 < x_2 < \cdots < x_k$ the numbers for which the statement holds. Let n be the number of digits of x_k. Obviously we have

$$(1) \qquad f(x_1 + a \cdot 10^n) + f(x_2 + a \cdot 10^n) = \cdots = f(x_k \cdot 10^n)$$

for all positive integers a, because if x_i is increased by $a \cdot 10^n$ then $d(x_i)$ is increased by $d(a)$.

Let $b = \overline{99\ldots9} = 10^n - 1$. We have $f(b) > f(x_k)$ because $b > x_k$ and $d(b) > d(x_k)$. Moreover, if $\alpha = f(b) - f(x_k)$ then from $b - x_k \geqslant 8 \cdot 10^{n-1} > 9n$, and from the fact that $d(b) > d(x_k)$ it follows that

$$(2) \qquad \alpha > 9n.$$

But $f(b+1) - f(b) = 2 - 9n$ thus $f(b+1) - f(x_k) = \alpha + 2 - 9n > 0$ (using (2)).

Let us consider a positive integer t such that $9t \geqslant \alpha + 2 - 9n > 9(t-1)$ and let us denote by y_i the number $\overline{99\ldots9} \cdot 10^n + x_i = (10^t - 1) \cdot 10^n + x_i$ and by c the number $(10^t - 1) \cdot 10^n + b$. It is easy to see that $f(y_1) = f(y_2) = \cdots = f(y_k)$ and $-9 < f(c+1) - f(y_k) = \alpha + 2 - 9n - 9t \leqslant 0$.

If $f(c+1) - f(y_k) = -2l$, $l \in \{0,1,2,3,4\}$, then $f(c+1+l) - f(y_k) = f(c+1) - f(y_k) + 2l = 0$, thus $y_{k+1} = c + 1 + l$ satisfies the requirements.

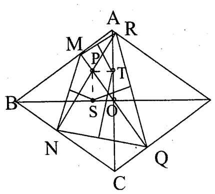
If $f(c+1) - f(y_k) = -2l + 1$, $l \in \{1,2,3,4\}$, then $f(c+1+l+4) - f(y_k) = f(c+1) - f(y_k) + 2(l+4) = -2l + 1 + 2l + 8 = 9$. Therefore, if we put another 9 in front of each of y_i-s and in front of c we obtain the numbers $z_i = \overline{9y_i}$ and $d = \overline{9c}$ for which we have $f(z_1) = f(z_2) = \cdots = f(z_k)$ and for which $f(d+1+l+4) - f(z_k) = f(c+1+l+4) - 9 - f(y_k) = 0$. Finally we observe that the first digit of $z_{k+1} = d+1+l+4$ is 1, and also the first digit of y_{k+1}, and obviously these numbers have at least two digits thus the statement is proved.

First team selection test for the Junior Balkan Mathematical Olympiad

PROBLEM 1. Consider a rhombus $ABCD$ with center O. A point P is given inside the rhombus, but not situated on the diagonals. Let M, N, Q, R be the projections of P on the sides $(AB), (BC), (CD), (DA)$, respectively. The perpendicular bisectors of the segments MN and RQ meet at S and the perpendicular bisectors of the segments NQ and MR meet at T.

Prove that P, S, T and O are the vertices of a rectangle.

Solution. First, observe that triangles RSN and QSM are congruent (S.S.S.), hence $\angle PMS = \angle PNS$ and $\angle PQS = \angle PQS$. It follows that P, S, M, N are concyclic and P, S, Q, R are concyclic.



On the other hand, as $\angle BNP + \angle BMP = 180°$, points B, N, S, M are concyclic, thus P, S, N, B, M are points on the circle $\mathcal{C}_1(O_1)$ of diameter BP. Likewise, points P, S, Q, D, R lie on the circle $\mathcal{C}_2(O_2)$ of diameter DP.

Since PS is the common chord of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$, lines PS and O_1O_2 are perpendicular. As O_1 and O_2 are the midpoints of the segments BP and DP, lines O_1O_2 and BD are parallel, so $PS \perp BD$ and then $PS \parallel AC$. Likewise, $PT \parallel BD$ and consequently $PS \perp PT$.

Furthermore, because O_1O_2 is middle line in the triangle PBD one find that S lies on the segment BD. Analogously, $T \in (AC)$. Thus, $PSOT$ is a rectangle.

PROBLEM 2. Consider the prime numbers $n_1 < n_2 < \cdots < n_{31}$. Prove that if 30 divides $n_1^4 + n_2^4 + \cdots + n_{31}^4$, then among these numbers one can find three consecutive primes.

Solution. Denote $S = n_1^4 + n_2^4 + \cdots + n_{31}^4$ and $A = \{n_1, n_2, \ldots, n_{31}\}$.

First, observe that $2 \in A$, otherwise all numbers n_i, $i = \overline{1,31}$ are odd and consequently S is odd; contradiction.

Then, $3 \in A$, else $n_i \equiv -1 (\bmod 3)$ and $n_i^4 \equiv 1 (\bmod 3)$ for all $i = \overline{1,n}$. It follows that $S \equiv 31 \equiv 1 (\bmod 3)$, contradiction.

Finally, we prove that $5 \in A$. Indeed, if not, then $n_i \equiv \pm 1 (\bmod 5)$ or $n_i \equiv \pm 2 (\bmod 5)$ for all $i = \overline{1,31}$. Consequently, $n_i^2 \equiv \pm 1 (\bmod 5)$ and $n_i^4 \equiv 1 (\bmod 5)$ for all $i = \overline{1,31}$. Thus, $S \equiv 31 \equiv 1 (\bmod 5)$, a contradiction.

PROBLEM 3. Let n be a positive integer. Prove that there are no positive integers x and y such as

$$\sqrt{n} + \sqrt{n+1} < \sqrt{x} + \sqrt{y} < \sqrt{4n+2}.$$

Solution. Assume that such numbers exist. By squaring,

$$(1) \qquad 2n + 1 + 2\sqrt{n^2+n} < x + y + 2\sqrt{xy} < 4n + 2.$$

Since $4n + 1 < x + y + 2\sqrt{xy} \leqslant 2(x+y)$, we obtain $x + y > 2n + \frac{1}{2}$. Numbers x and y are integers, so

$$x + y \geqslant 2n + 1.$$

Set $a = x + y - (2n+1) \geqslant 0$, where a is an integer. The second inequality from (1) gives $2\sqrt{xy} < 2n + 1 - a$, hence $4xy < (2n+1-a)^2$. Numbers $4xy$ and $2n + 1 - a$ are also integers, therefore $4xy \leqslant (2n+1-a)^2 - 1$ and then $2\sqrt{xy} \leqslant \sqrt{(2n+1-a)^2 - 1}$. From (1) we have

$$2\sqrt{n^2+n} < a + 2\sqrt{xy} \leqslant a + \sqrt{(2n+1-a)^2 - 1},$$

hence

$$(2) \qquad \sqrt{(2n+1)^2 - 1} - a \leqslant \sqrt{(2n+1-a)^2 - 1}.$$

As $x+y < 4n+2$, then $a = x+y-(2n+1) \leqslant 2n$ and so $a < \sqrt{(2n+1)^2 - 1}$. By squaring both sides of the relation (2) we obtain

$$(2n+1)^2 - 1 + a^2 - 2a\sqrt{(2n+1)^2 - 1} < (2n+1-a)^2 - 1$$
$$\Leftrightarrow -2a\sqrt{(2n+1)^2 - 1} < -2a(2n+1),$$

a contradiction.

PROBLEM 4. Show that one can color all the points of a plane using only two colors such that no line segment has all points of the same color.

Solution. Choose an arbitrary point A in the plane. Points located in the plane at a rational distance from A are colored in red, while the others are colored in blue. Consider an arbitrary segment PQ. We may assume that $AP < AQ$; if not, take instead of P another point of the line segment (PQ).

Recall that between two real numbers one can find a rational number q and an irational number r. The circles centered at A and having the radii q and r intersect the segment PQ at the points M and N respectively. It is obvious that M is colored in red and N in blue, so the claim is proved.

Second team selection test for the Junior Balkan Mathematical Olympiad

PROBLEM 5. Let a, b, c be positive real numbers with $abc = 1$. Prove that

$$1 + \frac{3}{a+b+c} \geqslant \frac{6}{ab+bc+ca}.$$

Solution. Setting $x = \frac{1}{a}$, $y = \frac{1}{b}$, $z = \frac{1}{c}$, we have $xyz = 1$. The inequality rewrites as

$$1 + \frac{3}{xy+yz+zx} \geqslant \frac{6}{x+y+z}.$$

Since $(x+y+z)^2 \geqslant 3(xy+yz+zx)$, it follows that

$$1 + \frac{3}{xy+yz+zx} \geqslant 1 + \frac{9}{(x+y+z)^2}.$$

It suffices to observe that

$$1 + \frac{9}{(x+y+z)^2} \geqslant \frac{6}{x+y+z},$$

which reduces to

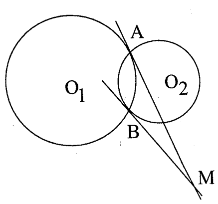
$$\left(1 - \frac{3}{x+y+z}\right)^2 \geqslant 0.$$

PROBLEM 6. Two circles $\mathcal{C}_1(O_1)$ and $\mathcal{C}_2(O_2)$ with distinct radii meat at points A and B. The tangent from A to $\mathcal{C}_1$ intersects the tangent from B to $\mathcal{C}_2$ at point M.

Show that both circles are seen from M under the same angle.

Solution. We have to prove that $2\angle O_1MA = 2\angle O_2BM$, which is equivalent to

(1) $$\frac{O_1A}{AM} = \frac{O_2B}{BM}.$$



The lenght of the common chord AB is equal to $2O_1A \cdot \sin\frac{1}{2}\overset{\frown}{AB} = 2O_1A \cdot \sin\angle BAM$, regardless if $\overset{\frown}{AB}$ is the small arc or the great arc $\overset{\frown}{AB}$. Similarly, $AB = 2O_2B \cdot \sin\angle ABM$, hence

(2) $$\frac{O_1A}{\sin\angle ABM} = \frac{O_2B}{\sin\angle BAM}.$$

By the Law of Sines in the triangle ABM we derive that

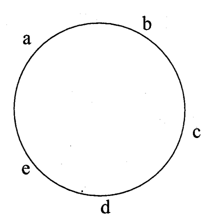
(3) $$\frac{MA}{\sin\angle ABM} = \frac{MB}{\sin\angle BAM}.$$

From the relations (2) and (3) we obtain $\frac{O_1A}{AM} = \frac{O_2B}{BM}$, as desired.

PROBLEM 7. Five real numbers of absolute values not greater than 1 and having the sum equal to 1 are written on the circumference of a circle.

Prove that one can choose three consecutively disposed numbers a, b, c, such that all the sums $a+b, b+c$ and $a+b+c$ are nonnegative.

Solution. First, we prove that at most two of the sums $a+b$, $b+c$, $c+d$, $d+e$ and $e+a$ can be negative.



Indeed, assume that two non-consecutive sums (say $a+b$ and $c+d$) are less than 0. Then $1 - e = (a+b) + (c+d) < 0$ and so $1 < e$, a contradiction. Thus, if three sums are negative, then two of them are not consecutive, which is false. Moreover, if two sums are negative, then these must be consecutive; in other words, at least three consecutive sums are nonnegative.

Let $a+b$, $b+c$, $c+d$ be greater than or equal to zero. If one of the sums $d+e$ or $e+a$ is negative, then $a+b+c = 1-(d+e)$ or $b+c+d = 1-(e+a)$ are at least 1, hence is positive and we are done.

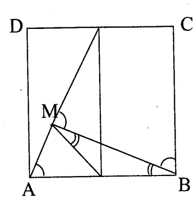
Finally, consider the case when all sums $a+b$, $b+c$, $c+d$, $d+e$, $e+a$ are positive. Suppose that $a+b+c < 0$; then $b > (a+b)+(b+c) > 0$. Thus, if $a+b+c$, $b+c+d$, $c+d+e$ are negative, then b, c, d are positive and we are done.

PROBLEM 8. Let E be the midpoint of the side CD of a square $ABCD$. Consider the point M inside the square such that

$$\angle MAB = \angle MBC = \angle BME = x.$$

Find the angle x.

Solution. Observe that $\angle MAB + \angle MBA = \angle MBC + \angle MBA = 90^\circ$, hence $\angle AMB = 90^\circ$.



Let F be the midpoint of the side AB. Then $MF = FA = FB = \frac{1}{2}AB$, so $\angle MBF = \angle MBF$. It follows that $\angle EMF = \angle EMB + \angle BMF = \angle MAB + \angle MBA = 90^\circ$.

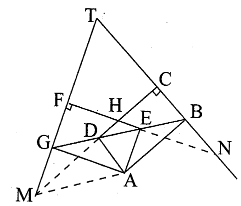
In the right triangle MEF, the leg MF is equal to $\frac{1}{2}EF$, hence $\angle MEF = 30^\circ$. We obtain $\angle MBF = \frac{1}{2}\angle MFA = \frac{1}{2}MEF = 15^\circ$ and $x = 75^\circ$.

Third team selection test for the Junior Balkan Mathematical Olympiad

PROBLEM 9. Suppose $ABCD$ and $AEFG$ are rectangles such that the points B, E, D, G are collinear (in this order). Let the lines BC and GF intersect at point T and let the lines DC and EF intersect at point H. Prove that points A, H and T are collinear.

Solution. Let the lines CD and FG intersect at M and let the lines BC and EF intersect at N. As $DHEA$ and $FHCT$ are cyclic quadrilaterals,

$$\angle FTC = 180^\circ - \angle FHC = \angle DAE \quad \text{and} \quad \angle DAH = \angle DEH.$$



Since $\angle DMG = 90^\circ - \angle FTC = 90^\circ - \angle DAE = \angle DAG$, it follows that the quadrilateral $ADGM$ is cyclic. Hence $\angle DAM = \angle FGE$ and consequently $\angle MAH = \angle DAM + \angle DAH = \angle FGE + \angle DEH = 90^\circ$. Likewise, $\angle NAH = 90^\circ$ and therefore points M, A, N are collinear.

In the triangle TMN, point H is the orthocenter. Thus A, H, T lie on the altitude of the triangle, as desired.

PROBLEM 10. Let a be a positive integer such that the number a^n has an odd number of digits in the decimal representation for all $n > 0$. Prove that the number a is an even power of 10.

Solution. Number a has an odd number of digits, hence $10^{2k} \leqslant a < 10^{2k+1}$ for some integer $k > 0$. It suffices to prove that $a = 10^{2k}$.

First, observe that $10^{4k} \leqslant a^2 < 10^{4k+2}$. Number a^2 has also an odd number of digits, hence $10^{2k} \leqslant a < 10^{2k+\frac{1}{2}}$. Next, $10^{8k} \leqslant a^4 < 10^{8k+2}$ and consequently $10^{2k} \leqslant a < 10^{2k+\frac{1}{4}}$. Inducting on n we obtain $10^{2k} \leqslant a < 10^{2k+\frac{1}{2n}}$ for all $n > 0$.

Assume by contradiction that $a \geqslant 10^{2k} + 1$. Then $10^{2k+\frac{1}{2n}} > 10^{2k} + 1 \Leftrightarrow 10^{2k} + \left(10^{\frac{1}{2n}} - 1\right) > 1 \Leftrightarrow 10^{\frac{1}{2n}} > 1 + \frac{1}{10^{2k}} \Leftrightarrow 10 > \left(1 + \frac{1}{10^{2k}}\right)^{2n}$, for all $n > 0$.

On the other hand, using Bernoulli inequality we find that

$$\left(1 + \frac{1}{10^{2k}}\right)^{2n} \geqslant 1 + \frac{2^n}{10^{2k}} \quad \text{for all } n > 0.$$

For sufficiently large n we have $1 + \frac{2^n}{10^{2k}} > 10$, a contradiction.

PROBLEM 11. A set of 2003 positive integers is given. Show that one can find two elements such that their sum is not a divisor of the sum of the other elements.

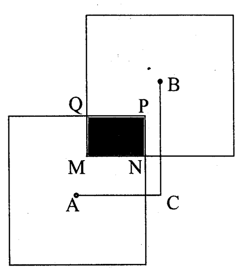
Solution. Let $a_1 < a_2 < \cdots < a_{2003}$ be the elements of the set. We prove the claim by contradiction. Numbers $a_1 + a_{2003}, a_2 + a_{2003}, \ldots, a_{2002} + a_{2003}$ divide the sum $S = a_1 + a_2 + \cdots + a_{2003}$, since $a + b \mid S - a - b$ if and only if $a + b \mid S$. Hence $S = k_i(a_i + a_{2003})$ for all $i = \overline{1, 2002}$, where k_i are integers.

Since $a_i + a_{2003} < S < 2003a_{2003} < 2003(a_i + a_{2003})$, it follows that $k_i \in \{2, 3, \ldots, 2002\}$ for all $i = \overline{1, 2002}$. By Pigeonhole principle, there is a pair of indices $i \neq j$ such that $k_i = k_j$, a contradiction.

PROBLEM 12. Two unit squares with parallel sides overlap by a rectangle of area $1/8$. Find the extreme values of the distance between the centers of the squares.

Solution. Let $MNPQ$ be the rectangle at the intersection of the unit squares with centers A and B. Set $MN = x$ and $PQ = y$, hence

$$xy = \frac{1}{8}, \quad x, y \in [0, 1].$$



The parallel from A to MN intersects the parallel from B to NP at point C. It is easy to observe that $AC = 1 - x$ and $BC = 1 - y$, so

$$\begin{aligned} AB^2 &= (1-x)^2 + (1-y)^2 = x^2 + y^2 - 2(x+y) + 2 \\ &= x^2 + 2xy + y^2 - 2(x+y) - \frac{1}{4} + 2 + (x+y)^2 - 2(x+y) + \frac{7}{4} \\ &= (x+y-1)^2 + \frac{3}{4}. \end{aligned}$$

It follows that the minimal value of the distance between the centers A and B is equal to $\frac{\sqrt{3}}{2}$, and it is obtained for $x + y = 1$, $xy = \frac{1}{8}$; i.e. $x = \frac{2+\sqrt{2}}{4}$, $y = \frac{2-\sqrt{2}}{4}$ or $x = \frac{2-\sqrt{2}}{4}$, $y = \frac{2+\sqrt{2}}{4}$.

To find the maximal value of AB observe that $0 \leqslant (1-x)(1-y) = 1 - x - y + xy = \frac{9}{8} - (x+y)$, so $x + y \leqslant \frac{9}{8}$. On the other hand, we have $x + y \geqslant 2\sqrt{xy} = \frac{1}{\sqrt{2}}$, therefore $\frac{1}{\sqrt{2}} - 1 \leqslant x + y - 1 \leqslant \frac{1}{8}$. As $\left(\frac{1}{8}\right)^2 \leqslant \left(\frac{1}{\sqrt{2}} - 1\right)^2$, we find that

$AB^2 \leqslant \frac{1}{2} - \sqrt{2} + 1\frac{3}{4} = \frac{9}{4} - \sqrt{2} = \left(2 - \frac{1}{\sqrt{2}}\right)^2$. Thus $AB \leqslant 2 - \frac{1}{\sqrt{2}}$, with equality when $x = y = \frac{1}{2\sqrt{2}}$.

Consequently, $\frac{\sqrt{3}}{2} \leqslant AB \leqslant 2 - \frac{1}{\sqrt{2}}$.

Part IV. SELECTION OF PROBLEMS SUMITTED IN MATHEMATICAL REGIONAL COMPETITIONS

IV.1. PROPOSED PROBLEMS

7^{th} GRADE

PROBLEM 1. There are 8 participants to a chess competition which take place in 7 rounds. The marks are distributed after well-known rules: 1 point for winning game, $\frac{1}{2}$ points for draw and 0 points for lost game.

(a) Show that after each of the first three rounds at least two participants have the same score.

(b) In the final ranking of participants there are no two with the same score. Find the least number of points the winer should have.

(Contest Unirea, Focşani; Lucian Buliga and Corneliu Savu)

PROBLEM 2. Let ABC be a triangle. For any interior point M of the triangle we denote by $s(M)$ the sum of distances of M to the sides AB, BC, CA. Show that if M, N are interior points of $\triangle ABC$ such that $s(M) = s(N)$ then for any point P of the segment MN one has $s(M) = s(P) = s(N)$.

(Contest Unirea, Focşani; Corneliu Savu)

PROBLEM 3. Find all integer numbers a, b, c, d such that $a^2 + b^2 = 2(c + d)$ and $c^2 + d^2 = 2(a + b)$.

(Contest Alexandru Myller, Iaşi; Gheorghe Iurea)

PROBLEM 4. Let $ABCD$ be a square and M, N be interior variable points on the sides BC, CD respectively, such that $MN = BM + DN$. Show that the angle NAM has constant measure.

(Contest Alexandru Myller, Iaşi; Gheorghe Iurea)

PROBLEM 5. Let a, b, c be nonnegative real numbers, not greater than 1 and such that $ab + bc + ac = 1$. Show that $a^2 + b^2 + c^2 \leqslant 2$.

(Contest Alexandru Myller, Iaşi; Mircea Becheanu)

8^{th} GRADE

PROBLEM 1. Show that the fractional part of the number $\sqrt{4n^2 + n}$ is not greater than 0.25.

(Contest Unirea, Focşani; Lucian Buliga and Corneliu Savu)

PROBLEM 2. Let k be a positive integer and $a = 3k^2 + 3k + 1$.

(i) Show that $2a$ and a^2 are sums of three perfect squares.

(ii) Show that if a is a divisor of a positive integer b and b is a sum of three perfect squares then any power b^n is a sum of three perfect squares.

(Contest Unirea, Focşani; Lucian Buliga and Corneliu Savu)

PROBLEM 3. Find positive integers x, y, z which verify conditions: $x + y \geqslant 2z$ and $x^2 + y^2 - 2z^2 = 8$.

(Contest Alexandru Myller, Iaşi; Adrian Zanoschi)

PROBLEM 4. A regular tetrahedron whose edge has length 1 is projected on a plane. Show that the area of the obtained polygon is not greater than $\frac{1}{2}$.

(Contest Alexandru Myller, Iaşi)

PROBLEM 5. Let $ABCD$ be a tetrahedron such that $AB = CD = a$, $AC = BD = b$, $AD = BC = c$ and let G_A, G_B, G_C, G_D be the centroids of the triangles BCD, CDA, DAB, ABC respectively.

Find the minimal length of a path on the faces of the tetrahedron and which passes through the points G_A, G_B, G_C and G_D.

(Contest Alexandru Myller, Iaşi)

PROBLEM 6. Let $n \geqslant 3$ be a positive integer. Show that it is possible to eliminate at most two numbers among the elements of the set $\{1, 2, \ldots, n\}$ such that the sum of remaining numbers is a perfect square.

(Contest Alexandru Myller, Iaşi; Mihai Bălună)

9^{th} GRADE

PROBLEM 1. Let $ABCD$ be a convex quadrilateral. Show that the following conditions are equivalent:

(a) $\frac{1}{AB} \cdot \overrightarrow{AB} + \frac{1}{BC} \cdot \overrightarrow{BC} + \frac{1}{CD} \cdot \overrightarrow{CD} + \frac{1}{DA} \cdot \overrightarrow{DA} = 0$.

(b) $ABCD$ is a parallelogram.

(Contest Unirea, Focşani)

PROBLEM 2. Find all positive integers n for which

$$\left\lfloor \frac{n^3+8n^2+1}{3n} \right\rfloor$$

is a prime number.

(Contest Unirea, Focşani; Gabriel Popa)

PROBLEM 3. Let ABC be a triangle, A_1, B_1, C_1 be points on the sides BC, CA, AB and A_2, B_2, C_2 be points on the sides B_1C_1, C_1A_1, A_1B_1 respectively. The following triples of lines are considered:

T_1: (AA_1, BB_1, CC_1)

T_2: (A_1A_2, B_1B_2, C_1C_2)

T_3: (AA_2, BB_2, CC_2).

Show that if the lines of two triples are concurrent then the lines of the third triple are concurrent too.

(Contest Unirea, Focşani; Dan Brânzei)

PROBLEM 4. Let a, b, c, d be positive numbers such that $abcd = 1$. Show that the following inequality holds:

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} \geqslant 4.$$

(Contest Unirea, Focşani; Andrei Nedelcu)

PROBLEM 5. Let $ABCD$ be a convex quadrilateral and O be an interior point. Let denote a, b, c, d, e, f the area of the triangle OAB, OBC, OCD, ODA, OAC, OBD, respectively. Show that $|ac - bd| = ef$.

(Contest Alexandru Myller, Iaşi; published by Al. Myller in Gazeta Matematică, 1949)

PROBLEM 6. (a) Show that there exist quadratic functions $f(x) = ax^2 + bx + c$ such that $f(f(k)) = k$, for any $k = 1, 2, 3$.

(b) Show that if f is a quadratic function as above then the numbers a, b, c cannot be all integers.

(Contest Alexandru Myller, Iaşi; Gheorghe Iurea)

PROBLEM 7. Let a, b, c, x, y, z be real numbers such that

$$x^2 + y^2 + z^2 = a + b + c = 1.$$

Show that $a(x+b) + b(y+c) + c(z+a) < 1$.

(Contest Nicolae Păun, Râmnicu Vâlcea)

PROBLEM 8. Let a, b be positive integers such that $a < b$ and $C = \{c_1, c_2, \dots, c_n\}$ a set of integer numbers such that $a \leqslant c_1 < c_2 < \cdots < c_n \leqslant b$ and $n > \frac{b-a+1}{2}$. Show that there are c_i, c_j in C such that $c_i + c_j = b + a$.

(Contest Nicolae Păun, Râmnicu Vâlcea; Radu Miculescu)

10th GRADE

PROBLEM 1. Given integer polynomial $f(X) = X^n + 2X^{n-1} + 3X^{n-2} + \cdots + nX + (n+1)$ and $\varepsilon = \cos\frac{2\pi}{n+2} + i\sin\frac{2\pi}{n+2}$, show that

$$f(\varepsilon)f(\varepsilon^2)\cdots f(\varepsilon^{n+1}) = (n+2)^n.$$

(Contest Alexandru Myller, Iaşi; Mihai Piticari)

PROBLEM 2. In a contest there are five examinations and the result is either passed or failed. Find the least number of contestants such that for any configuration of their responses there exist at least two contestants, say A and B, such that A passed all examinations in which B passed as well.

(Contest Alexandru Myller, Iaşi)

11th GRADE

PROBLEM 1. The sequence $(a_n)_{n\geqslant 1}$ is defined by conditions:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{1+a_n^2}{n}, \quad \forall n \geqslant 1.$$

(a) Show that $\lim\limits_{n\to\infty} a_n = 0$.

(b) Find the limit $\lim\limits_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n}$.

(Contest Unirea, Focşani; Laurenţiu Panaitopol)

PROBLEM 2. We are given a function $f : (0,\infty) \to \mathbf{R}$ and a nonconstant polynomial $P(X) \in \mathbf{R}[X]$ such that:

for all real $x > 0$, there exists and it is finite $\lim\limits_{t\to\infty} \frac{f(xt)}{P(t)}$.

Find the function $g : (0,\infty) \to \mathbf{R}$, defined by $g(x) = \lim\limits_{t\to\infty} \frac{f(xt)}{P(t)}$.

(Contest Unirea, Focşani; Laurenţiu Panaitopol)

PROBLEM 3. Let $A, B \in M_2(\mathbf{Z})$ be two matrices, with integer entries of dimension 2×2 such that $AB = BA$ and $\det A = \det B = 0$. Show that $\det(A^3 + B^3)$ is a cube of an integer number.

(Contest Unirea, Focşani; Mircea Becheanu)

PROBLEM 4. Let $A, B \in M_3(\mathbf{R})$ be two real matrices of dimension 3×3 with the following property:

any vector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ *which is a solution of the system* $AX = 0$ *is also a solution of the system* $BX = 0$.

Show that there exists a 3×3 matrix C such that $B = CA$.

(Contest Alexandru Myller, Iaşi; Mircea Becheanu)

PROBLEM 5. Show that for any positive integer n, there exist positive rational numbers $a_0 < a_1 < \cdots < a_n$ which satisfy the following conditions:

(a) $\frac{a_0}{0!} + \frac{a_1}{1!} + \cdots + \frac{a_n}{n!} = \frac{1}{n!}$;

(b) $a_0 + a_1 + \cdots + a_n < \frac{3}{2^n}$.

(Contest Alexandru Myller, Iaşi; Dorin Andrica)

PROBLEM 6. Find all derivable functions $f : [0, \infty) \to \mathbf{R}$ which have the properties:

(a) $f(0) = 0$;

(b) $f'(x) = \frac{1}{3}f'(\frac{x}{3}) + \frac{2}{3}f'(\frac{2x}{3})$, for all $x > 0$.

(Contest Alexandru Myller, Iaşi; Mihai Piticari)

12th GRADE

PROBLEM 1. Let K be a finite field with 27 elements. Show that there exists a, $a \in K$, such that $a^3 = a + 2$.

(Contest Unirea, Focşani; Mircea Becheanu)

PROBLEM 2. Let $f(X)$, $g(X)$ be irreducible rational polynomials and a, b complex numbers such that $f(a) = g(b) = 0$. Show that, if $a + b \in \mathbf{Q}$, then $f(X)$ and $g(X)$ have same degree.

(Contest Alexandru Myller, Iaşi; Bodgan Enescu)

PROBLEM 3. (a) Let $n > 0$ be an integer number. Show that

$$l_n = \lim_{t \to \infty} \int_1^t \frac{\sin x}{x^n}\, dx$$

exists and it is finite.

(b) Compute $\lim\limits_{n \to \infty} l_n$.

(Contest Alexandru Myller, Iaşi; Mihai Piticari)

IV.2. SOLUTIONS

7th GRADE

PROBLEM 1. There are 8 participants to a chess competition which take place in 7 rounds. The marks are distributed after well-known rules: 1 point for winning game, $\frac{1}{2}$ points for draw and 0 points for lost game.

(a) Show that after each of the first three rounds at least two participants have the same score.

(b) In the final ranking of participants there are no two with the same score. Find the least number of points the winer should have.

Solution. (a) The result is obvious after first two rounds. When we are after first three rounds, a player should obtain one of the following total number of points:

$$0,\ \frac{1}{2},\ 1,\ 1\frac{1}{2},\ 2,\ 2\frac{1}{2},\ 3.$$

Since there are 8 participants, by the Pigeonhole principle, there are two which have the same score.

(b) Let p be the final score of the winner in the described situation. The others, can have in a descending order, at most the following scores:

$$p - \frac{1}{2},\ p-1,\ p-\frac{3}{2},\ p-2,\ p-\frac{5}{2},\ p-3,\ p-\frac{7}{2}.$$

In this case, the total number of obtained points is $8p - 14$. Since 28 points are given after 7 rounds, it follows that $8p - 14 \geqslant 28$ and then $p \geqslant 5,5$ points.

The following table of scores shows that the case $p = 5,5$ can be obtained:

	1	2	3	4	5	6	7	8	Total
1		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	$5\frac{1}{2}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	5
3	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$4\frac{1}{2}$
4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	4
5	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	3
6	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$2\frac{1}{2}$
7	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	2
8	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$1\frac{1}{2}$

PROBLEM 2. Let ABC be a triangle. For any interior point M of the triangle we denote by $s(M)$ the sum of distances of M to the sides AB, BC, CA. Show that if M, N are interior points of ΔABC such that $s(M) = s(N)$ then for any point P of the segment MN one has $s(M) = s(P) = s(N)$.

Solution. Let M_A, M_B, M_C be the perpendicular projection of the point M on the sides BC, CA, AB, respectively. In the same way we introduce the points N_A, N_B, N_C and P_A, P_B, P_C. Let $\frac{PM}{PN} = k$. Then $PP_A = \frac{NN_A + kMM_A}{1+k}$, $PP_B = \frac{MM_B + kNN_B}{1+k}$, and $PP_C = \frac{MM_C + kNN_C}{1+k}$. By adding these equalities one obtains

$$PP_A + PP_B + PP_C = \sum MM_A = \sum NN_A = s(M).$$

PROBLEM 3. Find all integer numbers a, b, c, d such that $a^2 + b^2 = 2(c + d)$ and $c^2 + d^2 = 2(a + b)$.

Solution. We add the equalities and obtain:

$$(a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 = 4.$$

A solution can be given by $a - 1 = \pm 2$ and $b = c = d = 1$ which gives either $a = 3$, $b = c = d = 1$ or $a = 0$, $b = c = d = 1$. In both cases, the original conditions are not verified.

Another solution can be given by $a - 1 = \pm 1$, $b - 1 = \pm 1$, $c - 1 = \pm 1$, $d - 1 = \pm 1$. By checking all posibilities one obtains solutions: $(0,0,0,0)$, $(2,0,2,0)$, $(0,2,0,2)$, $(2,0,0,2)$, $(0,2,2,0)$ and $(2,2,2,2)$.

PROBLEM 4. Let $ABCD$ be a square and M, N be interior variable points on the sides BC, CD respectively, such that $MN = BM + DN$. Show that the angle NAM has constant measure.

Solution. Extend the side BC by a segment BP such that $BP = DN$. Then $\Delta ADN = \Delta ABP$. So, $AN = AP$ and $\angle DAN = \angle BAP$. It follows that the triangle MAP is right isosceles triangle.

Since $NM = MP$ it follows that AM is orthogonal on NP. So $\angle NAM = 45°$.

PROBLEM 5. Let a, b, c be nonnegative real numbers, not greater than 1 and such that $ab + bc + ac = 1$. Show that $a^2 + b^2 + c^2 \leqslant 2$.

Solution. Since $0 \leqslant a, b, c \leqslant 1$ we have $a^2 + b^2 + c^2 \leqslant a + b + c$. So, it is sufficient to prove $a + b + c \leqslant 2$. Since $(a-1)(b-1)(c-1) \leqslant 0$ we have:

$$a + b + c \leqslant 1 - abc + ab + bc + ca = 2 - abc \leqslant 2.$$

8th GRADE

PROBLEM 1. Show that the fractional part of the number $\sqrt{4n^2 + n}$ is not greater than 0.25.

Solution. From inequalities $4n^2 < 4n^2 + n < 4n^2 + n + 1$ one obtains $2n < \sqrt{4n^2 + n} < 2n + 1$. So, $\lfloor \sqrt{4n^2 + n} \rfloor = 2n$. We have to prove that $\sqrt{4n^2 + n} < 2n + 0.25$.

This is obvious, since by squaring the inequality one obtains: $4n^2 + n < 4n^2 + n + \frac{1}{16}$.

PROBLEM 2. Let k be a positive integer and $a = 3k^2 + 3k + 1$.

(i) Show that $2a$ and a^2 are sums of three perfect squares.

(ii) Show that if a is a divisor of a positive integer b and b is a sum of three perfect squares then any power b^n is a sum of three perfect squares.

Solution. (i) $2a = 6k^2 + 6k + 2 = (2k+1)^2 + (k+1)^2 + k^2$ and $a^2 = 9k^2 + 18k^3 + 15k^2 + 6k + 1 = (k^2 + k)^2 + (2k^2 + 3k + 1)^2 + k^2(2k+1) = a_1^2 + a_2^2 + a_3^2$.

(ii) Let $b = ca$. Then $b = b_1^2 + b_2^2 + b_3^2$ and $b^2 = c^2a^2 = c^2(a_1^2 + a_2^2 + a_3^2)$. To end the proof, we proceed as follows: for $n = 2p + 1$ we have $b^{2p+1} = (b^p)^2(b_1^2 + b_2^2 + b_3^2)$ and for $n = 2p + 2$, $b^{2p} = (b^p)^2b^2 = (bp)^2c^2(a_1^2 + a_2^2 + a_3^2)$.

PROBLEM 3. Find positive integers x, y, z which verify conditions: $x + y \geqslant 2z$ and $x^2 + y^2 - 2z^2 = 8$.

Solution. There are two possible cases:

Case I. $x \geqslant y \geqslant z$.

We denote $x - z = a \geqslant 0$, $y - z = b \geqslant 0$, $a \geqslant b$. One obtains the equation $2z(a + b) + a^2 + b^2 = 8$. When $z \geqslant 3$, there are no solutions. For $z = 2$, we get $(a+2)^2+(b+2)^2 = 16$, which again has no solution. When $z = 1$ we obtain solutions $(x, y, z) = (3, 1, 1)$ or $(x, y, z) = (1, 3, 1)$. When $z = 0$, $a^2 + b^2 = 8$ and we get the solution $(x, y, z) = (2, 2, 0)$.

Case II. $x \geqslant z \geqslant y$.

Note again $x - z = a$, $y - z = b$ and obtain the solution $(x, y, z) = (n+2, n-2, n)$ or $(x, y, z) = (n - 2, n + 2, n)$.

PROBLEM 4. A regular tetrahedron whose edge has length 1 is projected on a plane. Show that the area of the obtained polygon is not greater than $\frac{1}{2}$.

Solution. The obtained polygon is a triangle or a quadrilateral. The maximal area of a triangle is $\frac{\sqrt{3}}{4}$. The area of the quadrilateral is $\frac{1}{2}d_1d_2\sin\varphi$, where d_1, d_2 are the lengths of its diagonals and φ is the angle between them. Therefore, the area of the quadrilateral is not greater than $\frac{1}{2}$.

PROBLEM 5. Let $ABCD$ be a tetrahedron such that $AB = CD = a$, $AC = BD = b$, $AD = BC = c$ and let G_A, G_B, G_C, G_D be the centroids of the triangles BCD, CDA, DAB, ABC respectively.

Find the minimal length of a path on the faces of the tetrahedron and which passes through the points G_A, G_B, G_C and G_D.

Solution. The tetrahedron has congruent faces. Therefore, the sum of angles is each vertex of the tetrahedron is 180°. When we unfold it on a plane we obtain either a triangle or a parallelogram. Therefore, we can choose the path $G_DG_AG_BG_C$ whose length is $\frac{2}{3}(m_1 + m_2 + m_3)$ where m_1, m_2, m_3 are the lengths of medians in the triangle ABC.

PROBLEM 6. Let $n \geqslant 3$ be a positive integer. Show that it is possible to eliminate at most two numbers among the elements of the set $\{1, 2, \ldots, n\}$ such that the sum of remaining numbers is a perfect square.

Solution. Let $m = \left\lfloor \sqrt{\frac{n(n+1)}{2}} \right\rfloor$. From $m^2 \leqslant \frac{n(n+1)}{2} < (m + 1)^2$ we obtain $\frac{n(n+1)}{2} - m^2 < (m + 1)^2 - m^2 = 2m + 1$. Therefore, we have:

$$\frac{n(n+1)}{2} - m^2 \leqslant 2m \leqslant \sqrt{2n^2 + 2n} \leqslant 2n - 1.$$

Since, any number k, $k \leqslant 2n - 1$ can be obtained by adding at most two numbers from $\{1, 2, \ldots, n\}$, we obtain the result.

9th GRADE

PROBLEM 1. Let $ABCD$ be a convex quadrilateral. Show that the following conditions are equivalent:

(a) $\frac{1}{AB} \cdot \overrightarrow{AB} + \frac{1}{BC} \cdot \overrightarrow{BC} + \frac{1}{CD} \cdot \overrightarrow{CD} + \frac{1}{DA} \cdot \overrightarrow{DA} = 0$.

(b) $ABCD$ is a parallelogram.

Solution. The vectors $a = \frac{1}{AB} \cdot \overrightarrow{AB}$, $b = \frac{1}{BC} \cdot \overrightarrow{BC}$, $c = \frac{1}{CD} \cdot \overrightarrow{CD}$ and $d = \frac{1}{DA}\overrightarrow{DA}$ are unit vectors and have the same directions as the sides of the quadrilateral, respectively. Since $a + b + c + d = 0$, it follows that a, b, c, d are the sides of a rhombus. Then $ABCD$ is a parallelogram.

PROBLEM 2. Find all positive integers n for which

$$\left\lfloor \frac{n^3 + 8n^2 + 1}{3n} \right\rfloor$$

is a prime number.

Solution. Since the denominator is divisible by 3 we consider the division of n by 3.

If $n = 3p$, then

$$\left\lfloor \frac{n^3 + 8n^2 + 1}{3n} \right\rfloor = 3p^2 + 8p = p(3p + 8);$$

this is a prime number only for $p = 1$.

When $n = 3p + 1$,

$$\left\lfloor \frac{n^3 + 8n^2 + 1}{3n} \right\rfloor = (3p + 1)(p + 3),$$

which is a prime number only for $p = 0$.

When $n = 3p + 2$,

$$\left\lfloor \frac{n^3 + 8n^2 + 1}{3n} \right\rfloor = \left\lfloor \frac{27p^3 + 126p^2 + 132p + 41}{9p + 6} \right\rfloor = 3p^2 + 12p + 6 = 3(p^2 + 4p + 2)$$

is a composite number.

PROBLEM 3. Let ABC be a triangle, A_1, B_1, C_1 be points on the sides BC, CA, AB and A_2, B_2, C_2 be points on the sides B_1C_1, C_1A_1, A_1B_1 respectively. The following triples of lines are considered:

T_1: (AA_1, BB_1, CC_1)

T_2: (A_1A_2, B_1B_2, C_1C_2)

T_3: (AA_2, BB_2, CC_2).

Show that if the lines of two triples are concurrent then the lines of the third triple are concurrent too.

Solution. We introduce the intersection points A_3, B_3, C_3 to be $A_3 = AA_2 \cap BC$, etc. Let assume that triples T_1, T_2 are concurrent. By Ceva theorem, one has

$$\frac{A_1B}{A_1C} \cdot \frac{B_1C}{B_1A} \cdot \frac{C_1A}{C_1B} = 1 \quad \text{and} \quad \frac{A_2B_1}{A_2C_1} \cdot \frac{B_2C_1}{B_2A_1} \cdot \frac{C_2A_1}{C_2B_1} = 1.$$

Then $\frac{A_3C}{A_3B} = \frac{A_2B_1}{A_2C_1} \cdot \frac{AC_1}{AB_1} \cdot \frac{AC}{AB}$. Similar expressions can be obtained for $\frac{B_3A}{B_3C}$ and $\frac{C_3B}{C_3A}$. Let denote:

$$\alpha = \prod \frac{A_3C}{A_3B}, \quad \beta = \frac{C_1A}{C_1B}, \quad \gamma = \frac{B_1A_2}{C_1A_2}$$

where the products are taken by cyclic permutations. When multiply, we obtain $\gamma = \alpha\beta$. This proves the result.

PROBLEM 4. Let a, b, c, d be positive numbers such that $abcd = 1$. Show that the following inequality holds:

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} \geqslant 4.$$

Solution. We replace $cd = \frac{1}{ab}$ and $da = \frac{1}{bc}$. The expression turn out to be

$$\begin{aligned} E &= \frac{1+ab}{1+a} + \frac{1+ab}{ab+abc} + \frac{1+bc}{1+b} + \frac{1+bc}{bc+bcd} \\ &= (1+ab)\left(\frac{1}{1+a} + \frac{1}{ab+abc}\right) + (1+bc)\left(\frac{1}{1+b} + \frac{1}{bc+bcd}\right). \end{aligned}$$

Using the inequality $\frac{1}{x} + \frac{1}{y} \geqslant \frac{4}{x+y}$ we obtain:

$$\begin{aligned} E &= (1+ab)\frac{4}{1+a+ab+abc} + (1+bc)\frac{4}{1+a+bc+bcd} \\ &= 4\left(\frac{1+ab}{1+a+ab+abc} + \frac{1+bc}{1+a+bc+bcd}\right) \\ &= 4\left(\frac{1+ab}{1+a+ab+abc} + \frac{a+abc}{a+ab+abc+abcd}\right) = 4. \end{aligned}$$

PROBLEM 5. Let $ABCD$ be a convex quadrilateral and O be an interior point. Let denote a, b, c, d, e, f the area of the triangle OAB, OBC, OCD, ODA, OAC, OBD, respectively. Show that $|ac - bd| = ef$.

Solution. Let I be the intersection point of diagonals AC and BD. We may assume that I is inside the triangle BIC.

We denote by $\alpha, \beta, \gamma, \delta$ the angles AOB, BOC, COD, DOA, respectively. Then, we have to prove the equality:

$$ac = bd + ef.$$

It is equivalent to:

$$\begin{aligned} OA \cdot OB \sin\alpha \cdot OD \cdot OC \cdot \sin\gamma &= OB \cdot OC \sin\beta \cdot OD \cdot OA \cdot \sin\delta \\ &\quad + OD \cdot OB \sin(\alpha+\delta) \cdot OA \cdot OC \cdot \sin(\delta+\gamma). \end{aligned}$$

This is succesively equivalent to:

$$\begin{aligned} &\sin\alpha \sin\gamma = \sin\beta \sin\delta + \sin(\alpha-\delta)\sin(\delta-\gamma) \\ &\Leftrightarrow \cos(\alpha-\gamma) - \cos(\alpha+\gamma) = \cos(\beta-\delta) - \cos(\beta+\delta) + 2\sin(\alpha+\delta)\sin(\delta+\gamma) \\ &\Leftrightarrow \cos(\alpha-\gamma) - \cos(\beta-\delta) = 2\sin(\alpha+\delta)\sin(\delta+\gamma) \\ &\Leftrightarrow -2\sin\frac{\alpha-\gamma-\beta+\delta}{2}\sin\frac{\alpha-\gamma+\beta-\delta}{2} = 2\sin(\alpha+\delta)\sin(\delta+\gamma) \\ &\Leftrightarrow -\sin(\pi-(\gamma+\beta))\sin(\pi-(\gamma+\delta)) = \sin(\alpha+\delta\sin(\delta+\gamma) \\ &\Leftrightarrow \sin(\gamma+\beta) = \sin(\alpha+\delta). \end{aligned}$$

Last equality is valid because $\gamma + \beta = 2\pi - (\alpha + \delta)$.

PROBLEM 6. (a) Show that there exist quadratic functions $f(x) = ax^2 + bx + c$ such that $f(f(k)) = k$, for any $k = 1, 2, 3$.

(b) Show that f is a quadratic function as above then the numbers a, b, c cannot be all integers.

Solution. (a) We look for a function f such that $f(1) = 1$, $f(2) = 2$, $f(3) = 3$. By solving the obtained system one obtains: $a = -\frac{3}{2}$, $b = \frac{13}{2}$, $c = -4$.

(b) Assume by contrary that such a function exists, where $a, b, c \in \mathbf{Z}$. Since $x - y \mid f(x) - f(y)$, for all distinct $x, y \in \mathbf{Z}$ we obtain that $f(1) - f(2)$, $f(2) - f(3)$, $f(1) - f(3)$ are in the set $\{\pm1, \pm2\}$. But $(f(1)-f(2)) + (f(2)-f(3)) + (f(3)-f(1)) = 0$. So, $f(1) - f(2) = \pm1$, $f(2) - f(3) = \pm1$ and $f(3) - f(1) = \mp2$. After replacing this, we obtain $a = 0$, which contradicts f is a quadratic function.

PROBLEM 7. Let a, b, c, x, y, z be real numbers such that

$$x^2 + y^2 + z^2 = a + b + c = 1.$$

Show that $a(x+b) + b(y+c) + c(z+a) < 1$.

(Contest Nicolae Păun, Râmnicu Vâlcea)

Solution. Let denote $p = a^2 + b^2 + c^2$. Then one has:

$$2(ab + bc + ca) = 1 - p \quad \text{and} \quad (ax + by + cz)^2 \leqslant p.$$

It follows that:

$$a(x+b) + b(y+c) + c(z+a) \leqslant \frac{1}{2}(1-p) + \sqrt{p} = 1 - \left(\frac{1-\sqrt{p}}{2}\right)^2 < 1.$$

PROBLEM 8. Let a, b be positive integers such that $a < b$ and $C = \{x_1, x_2, \ldots, x_n\}$ a set of integer numbers such that $a \leqslant x_1 < x_2 < \cdots < x_n \leqslant b$ and $n > \frac{b-a+1}{2}$. Show that there are x_i, x_j in C such that $x_i + x_j = b + a$.

Solution. Let $C = \{x_1, x_2, \ldots, x_n\}$ such that $n > \frac{b-a+1}{2}$. Consider also the set $C' = \{a+b-x_1, \ldots, a+b-x_n\}$. One has $|C \cap C'| = |C| + |C'| - |C \cup C'|$. Since $|C| = |C'| = n$, $n > \frac{b-a+1}{2}$ and $C \cup C' \subset [a,b] \cap \mathbf{N}$ one obtains easily that $|C \cap C'| \geqslant 1$. This shows that there exist $x_i, x_j \in C$ such that $a + b - x_i = x_j$, that is $a + b = x_i + x_j$.

10th GRADE

PROBLEM 1. Given integer polynomial $f(X) = X^n + 2X^{n-1} + 3X^{n-2} + \cdots + nX + (n+1)$ and $\varepsilon = \cos\frac{2\pi}{n+2} + \mathrm{i}\sin\frac{2\pi}{n+2}$, show that

$$f(\varepsilon)f(\varepsilon^2)\cdots f(\varepsilon^{n+1}) = (n+2)^n.$$

Solution. Let $g(X) = X^{n+1} + X^n + \cdots + X + 1$ be the polynomial whose roots are $\varepsilon, \varepsilon^2, \ldots, \varepsilon^{n+1}$. We mention the key equality: $g(X) = (X-1)f(X) + n + 2$. From it we obtain:

$$g(\varepsilon^k) = (\varepsilon^k - 1)f(\varepsilon^k) + n + 2, \quad \forall\, k = 1, \ldots, n+1.$$

After writting them under the form

$$(1 - \varepsilon^k)f(\varepsilon^k) = n + 1, \quad \forall\, k = 1, \ldots, n+1$$

and by multiplication of all these equalities, we obtain:

$$(1-\varepsilon)(1-\varepsilon^2)\cdots(1-\varepsilon^{n+1})\prod_{k=1}^{n+1} f(\varepsilon^k) = (n+2)^{n+1}.$$

Since $(1-\varepsilon)(1-\varepsilon^2)\cdots(1-\varepsilon^{n+1}) = g(1) = n+2$, the result is done.

PROBLEM 2. In a contest there are five examinations and the result is either passed or failed. Find the least number of contestants such that for any configuration of their responses there exist at least two contestants, say A and B, such that A passed all examinations in which B passed as well.

Solution. Let a, b, c, d, e be the five examinations. In the case of at most 10 contestants it is possible that each contestant has passed exactly two examinations. In this case, it is possible that two distinct contestants pass distinct examinations, since $\binom{5}{2} = 10$.

When assume that there are at least 11 contestants we may consider the following ten families of subsets of the set $\{a, b, c, d, e\}$:

$$\{\emptyset, a, ab, abc, abcd, abcde\}; \quad \{b, bc, bcd, bcde\}; \quad \{c, ac, acd, acde\}; \quad \{d, cd, cde\};$$

$$\{e, de, ade\}; \quad \{ad, abd\}; \quad \{ac, ace, abce\}; \quad \{ae, abe\}; \quad \{be, bde\}; \quad \{ce, bce\}.$$

Since there exist two contestants which passed the same examinations, the result follows.

11th GRADE

PROBLEM 1. The sequence $(a_n)_{n\geqslant 1}$ is defined by conditions:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{1 + a_n^2}{n}, \quad \forall\, n \geqslant 1.$$

(a) Show that $\lim\limits_{n\to\infty} a_n = 0$.

(b) Find the limit $\lim\limits_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}$.

Solution. (a) By induction, we show that $a_n < 3$, $\forall\, n$. We have $a_1 = a_2 = 1$, $a_3 = \frac{2}{3}$ and from $n > 3$ and $a_n < 3$ we get:

$$a_{n+1} = \frac{1 + a_n^2}{n} < \frac{10}{n} < 3.$$

(b) The following inequalities hold:

$$\frac{1}{n-1} < a_n < \frac{1}{n-1} + 100\left(\frac{1}{n-1} - \frac{1}{n}\right), \quad \forall n \geqslant 1.$$

After summing up to n we obtain:

$$1 + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) < \sum_{k=1}^{n} a_k < \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) + 101.$$

Since

$$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{\ln n} = 1$$

we obtain

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{\ln n} = 1.$$

PROBLEM 2. We are given a function $f : (0, \infty) \to \mathbf{R}$ and a nonconstant polynomial $P(X) \in \mathbf{R}[X]$ such that:

for all real $x > 0$, there exists and it is finite $\lim\limits_{t\to\infty} \frac{f(xt)}{P(t)}$.

Find the function $g : (0, \infty) \to \mathbf{R}$, defined by $g(x) = \lim\limits_{t\to\infty} \frac{f(xt)}{P(t)}$.

Solution. Let $P(X) = a_n X^n + \cdots + a_1 X + a_0$ the algebraic expression of the polynomial. Let $xt = y$ a new variable, where $t > 0$ is a parameter and x is arbitrary, but fixed. Since $t = \frac{x}{y}$ we get

$$g(x) = \lim_{y\to\infty} \frac{f(y)}{\sum a_k \frac{y^k}{x^k}} = \lim_{y\to\infty} \frac{f(y)}{y^n} \cdot \lim_{y\to\infty} \frac{x^n}{a_n + a_{n-1}\frac{x}{y} + \cdots + a_0 \frac{x^n}{y^n}}.$$

Therefore, it exists $\lim\limits_{y\to\infty} \frac{f(y)}{y^n} = \ell$. Hence, $g(x) = \ell \frac{x^n}{a_n} = bx^n$ where $b = \frac{\ell}{a_n}$.

PROBLEM 3. Let $A, B \in M_2(\mathbf{Z})$ be two matrices, with integer entries of dimension 2×2 such that $AB = BA$ and $\det A = \det B = 0$. Show that $\det(A^3 + B^3)$ is a cube of an integer number.

Solution. Let x be a real variable. Then, $\det(A+xB)$ is an integer polynomial of the form:

$$P(x) = \det(A + xB) = \det Bx^2 + mx + \det A.$$

Since $\det A = \det B = 0$, we get $\det(A + xB) = mx = P(x)$.

From the algebraic decomposition:

$$A^3 + B^3 = (A + B)(A + \varepsilon B)(A + \varepsilon^2 B),$$

where $\varepsilon = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}$, we obtain:

$$\det(A^3 + B^3) = P(1) \cdot P(\varepsilon) \cdot P(\varepsilon^2) = m \cdot m\varepsilon \cdot m\varepsilon^2 = m^3.$$

PROBLEM 4. Let $A, B \in M_3(\mathbf{R})$ two real matrices of dimension 3×3 with the following property:

any vector $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ *which is a solution of the system* $AX = 0$ *is also a solution of the system* $BX = 0$.

Show that there exists a 3×3 matrix C such that $B = CA$.

Solution. If A is nonsingular matrix, we may take $C = BA^{-1}$. So, we have to consider only the case $\det A = 0$. The condition $B = CA$ means: the columns of B are linear combination of columns of A, when take them as vectors in $\mathbf{R}^3$.

Let a_1, a_2, a_3 be the columns of A and b_1, b_2, b_3 be the columns of B. Since $\operatorname{rank} A < 3$, there exists real numbers, not all zeros, such that

(1) $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$.

By the hypothesis, the columns of B satisfy the same relation:

(2) $\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = 0$.

Condition (1) represents the equation of either a plane or a line in $\mathbf{R}^3$. Since vectors b_1, b_2, b_3 belong to that plane (or line) they are a linear combination of a base of it. But a base can be chosen from the columns of A.

PROBLEM 5. Show that for any positive integer n, there exist positive rational numbers $a_0 < a_1 < \cdots < a_n$ which satisfy the following conditions:

(a) $\frac{a_0}{0!} + \frac{a_1}{1!} + \cdots + \frac{a_n}{n!} = \frac{1}{n!}$;

(b) $a_0 + a_1 + \cdots + a_n < \frac{3}{2^n}$.

Solution. We have the well-known formula:

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} = 2^n.$$

From it we obtain:

$$\sum_{k=0}^{n} \frac{1}{k!} \frac{1}{2^n (n-k)!} = \frac{1}{n!}.$$

So, we may take $a_k = \frac{1}{2^n (n-k)!}$, and obviously one has $a_0 < a_1 < \cdots < a_n$.

(b) Moreover,

$$\sum_{k=0}^{n} a_k = \frac{1}{2^n} \sum_{k=0}^{n} \frac{1}{(n-k)!} = \frac{1}{2^n}\left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) < \frac{1}{2^n} \cdot \mathrm{e} < \frac{3}{2^n}.$$

PROBLEM 6. Find all derivable functions $f : [0, \infty) \to \mathbf{R}$ which have the properties:

(a) $f(0) = 0$;

(b) $f'(x) = \frac{1}{3}f'(\frac{x}{3}) + \frac{2}{3}f'(\frac{2x}{3})$, all $x > 0$.

Solution. We have $f(x) = f(\frac{2}{3}) + f(\frac{2x}{3})$, $\forall\, x > 0$. Consider the function:

$$G(x) = \begin{cases} \frac{f(x)}{x}, & x > 0, \\ f'(0), & x = 0. \end{cases}$$

The function G is continuous on $[0, \infty)$ and $G(x) = \frac{1}{3}G(\frac{x}{3}) + \frac{2}{3}G(\frac{2x}{3})$.

Let $a > 0$ and denote $M = \sup\limits_{x \in [0,a]} G(x)$. There exists x_0, $0 \leqslant x_0 \leqslant a$, such that $G(x_0) = M$. Since $G(x) = \frac{1}{3}G(x) + \frac{2}{3}G(\frac{2x}{3})$ it follows that $M = \frac{1}{3}G(\frac{x_0}{3}) + \frac{2}{3}G(\frac{2x_0}{3})$, so $G(\frac{x_0}{3}) = M$.

In a similar way, we obtain $G(\frac{x_0}{3^n}) = M$, $\forall\, n$.

We deduce that $M = f'(0)$. In the same way, one can deduce that $f'(0) = \inf\limits_{x \in [0,a]} G(x)$. So, $G(x) = f'(0)$ and $f(2) = f'(0) \cdot x$, $\forall\, x$.

12^{th} GRADE

PROBLEM 1. Let K be a finite field with 27 elements. Show that there exists a, $a \in K$, such that $a^3 = a + 2$.

Solution. The field K is commutative of characteristic 3. The group $G = K^*$ is abelian of order 26. So, for any $x \in K^*$, we have $x^{26} - 1 = 0$. The polynomial $X^{26} - 1$ has the divisor $X^3 - X + 1$, so among all solutions $x \in K^*$ of the equation $x^{26} - 1 = 0$, there are a such that $a^3 = a - 1 = a + 2$.

PROBLEM 2. Let $f(X)$, $g(X)$ be irreducible rational polynomials and a, b complex numbers such that $f(a) = g(b) = 0$. Show that, if $a + b \in \mathbf{Q}$, then $f(X)$ and $g(X)$ have same degree.

Solution. Let $a + b = r \in \mathbf{Q}$. Then $f(a) = f(r - b) = 0$. The polynomial $h(X) = f(r - X)$ has rational coefficients and $\deg h = \deg f$. Moreover, $h(X)$ is irreducible in $\mathbf{Q}[X]$.

Since $h(b) = g(b) = 0$ and they are irreducible, it follows that $h(X) = \alpha g(X)$; so $\deg f = \deg g$.

PROBLEM 3. Let $n > 0$ be an integer number.

(a) Show that

$$\ell_n = \lim_{t \to \infty} \int_1^t \frac{\sin x}{x^n}\,\mathrm{d}x$$

exists and it is finite.

(b) Compute $\lim\limits_{n \to \infty} \ell_n$.

Solution. (a) We consider $F_n(t) = \int_1^t \frac{\sin x + 1}{x^n}\,\mathrm{d}x$. The function $F_n(t)$ is monotonic increasing, so $\lim\limits_{t \to \infty} F_n(t)$ exists.

From the inequalities:

$$F_n(t) \leqslant \int_1^t \frac{2}{x^n}\,\mathrm{d}x \leqslant \frac{2}{n-1}, \quad \forall n > 0$$

we obtain that $\lim\limits_{t \to \infty} F_n(t) < \infty$. Since $\lim\limits_{t \to \infty} \int_1^t \frac{-1}{x^n} < \infty$ it follows that $\ell_n < \infty$.

(b) From $\int_1^t \frac{-1}{x^n}\,\mathrm{d}x \leqslant \int_1^t \frac{\sin x + 1}{x^n}\,\mathrm{d}x \leqslant \int_1^t \frac{1}{x^n}\,\mathrm{d}x$ we obtain $-\frac{1}{n-1} \leqslant \ell_n \leqslant \frac{1}{n-1}$. So, $\lim\limits_{t \to \infty} \ell_n = 0$.

Part V. THE INSTITUTE OF MATHEMATICS TEST FOR HIGH-SCHOOL STUDENTS

Bucharest, January 19, 2003

PROBLEM 1. Prove that the interior of a convex pentagon $ABCDE$, having all sides of equal length, cannot be entirely covered by the open discs having the sides of the pentagon as diameters.

* * *

Solution. Let us denote by $2R$ the side length of $ABCDE$. It follows directly from the Pigeonhole principle that there are two consecutive angles of the pentagon greater than $60°$. Suppose that these angles are $\angle EAB$ and $\angle ABC$. It follows that BE and AC are greater than $2R$.

Let M be the middle point of the segment EC. The point M is on the semicircle of diameter DE and DC, therefore it lies in their exterior. We shall prove that M also lies in the exterior of the semicircle of diameter AE. Indeed, $MF = \frac{AC}{2} > R$, where F is the middle point of the segment AC. The same follows for the semicircle of diameter BC.

All we have left to prove is that M is in the exterior of the semicircle of diameter AB. Suppose otherwise, which means that $\angle AMB > 90°$. Then AB is the greatest side in the triangle AMB, thus $AM < 2R$. But from

$$EM = \frac{1}{2}EC < \frac{1}{2}(ED + DC) = 2R,$$

it follows that $EA = 2R > EM$ and $EA > AM$ and thus $\angle EMA > 60°$. In the same way $\angle BMC > 60°$, therefore $180° < \angle EMA + \angle AMB + \angle BMC$, which is a contradiction.

PROBLEM 2. Prove that in any triangle ABC the following inequality holds:

$$p\sqrt{3} \geqslant l_a + l_b + l_c,$$

where l_a, l_b, l_c are the lengths of the interior angle bisectors of $\angle BAC$, $\angle ABC$ and $\angle ACB$ respectively, and p is the semi-perimeter of the triangle ABC.

Valentin Vornicu

Solution. We will use the following:

LEMMA. In any such triangle ΔABC one has $l_a^2 \leqslant p(p-a)$.

Proof of Lemma. Indeed because

$$l_a^2 = \frac{4bc}{(b+c)^2}p(p-a)$$

using the AM-GM inequality for b and c the claim is obvious.

Now, using the Cauchy-Schwartz inequality and the lemma previously established one gets:

$$(l_a + l_b + l_c)^2 \leqslant 3\left(l_a^2 + l_b^2 + l_c^2\right) \leqslant 3\left[3p^2 - p(a+b+c)\right] = 3p^2$$

which gives us exactly the required inequality.

PROBLEM 3. Let Γ be the circumcircle of a triangle ABC and I the incircle of the same triangle. Consider the circle tangent to the sides CA, CB respectively in D, E and interior tangent to the circle Γ. Prove that I is the middle point of the segment DE.

IMO Short List 1993

Solution. It is obvious that if we succeed in proving that I lies of the segment DE then the problem is solved, because the triangle DEF is isosceles, and CI is an angle bisector which means that it is a median line in the same time.

Let us denote by x and y the lengths of the segments BE respectively AD. From Casey's theorem applied to the points A, B, C and the circle Ω we obtain that:

$$xb + ya = (a - x)c.$$

But $CE = CD$ thus $a - x = b - y \Rightarrow y = b - a + x$. Solving the above system we obtain:

$$x = \frac{a(p-b)}{p} \quad \text{and} \quad y = \frac{b(p-a)}{p}. \tag{1}$$

The fact that I lies on DE can be expressed using the transversal theorem that

$$I \in DE \Leftrightarrow AC' \cdot \frac{BE}{FC} + BC' \cdot \frac{AD}{DC} = AB \cdot \frac{C'I}{IC}$$

where $C' = CI \cap AB$. We know from the bisector theorem that

$$\frac{C'I}{IC} = \frac{c}{a+b}$$

therefore we want to prove that

$$\frac{cb}{b+a}\cdot\frac{x}{a-x}+\frac{ca}{a+b}\cdot\frac{y}{b-y}=\frac{c^2}{a+b}$$
$$\Leftrightarrow \frac{bx}{a-x}+\frac{ay}{b-y}=c$$
$$\Leftrightarrow \frac{b\frac{a(p-b)}{p}}{a-\frac{a(p-b)}{p}}+\frac{a\frac{b(p-a)}{p}}{b-\frac{b(p-a)}{p}}=c$$
$$\Leftrightarrow \frac{ba(p-b)}{pa-ap+ab}+\frac{ba(p-a)}{pb-bp+ba}=c$$

the last equality being obviously true.

PROBLEM 4. On an island there are n, $n \geqslant 2$, natives. Any two natives are either friends, either enemies. Each native is ordered by the tribes' chief to wear a necklace with colored stones such that any pair of native friends to each have at least a stone of the same color and any pair of native enemies to have no stone of the same color in their necklaces. It is possible that some of the necklaces are without any stones.

What is the minimal number of required colors so that the natives can paint the stones in order to respect their chief's order, no matter what the relationships between the natives are?

Belarussian Olympiad

Solution. The case in which n is even and the natives divide themselves into 2 equal groups, in each group being only relative enemies and each native being friend with each native from the other group, suggests the answer $n = \left\lfloor \frac{n^2}{4} \right\rfloor$.

We shall prove that the above claim by induction from n to $n+2$. The verification step for $n = 1$ and $n = 2$ is easy.

Let us suppose that the statement holds for n natives and let us prove it for $n+2$. If all the natives are enemies there are no stones required. Thus let us consider two friends A and B. The need at most a new color. The other need $\left\lfloor \frac{n^2}{4} \right\rfloor$ types of stones, from our supposition. If C is one of these natives, he can be friends with both A and B, or enemy of both, or just a friend of one and enemy of the other one. In each of the cases at most one new color is required. But C was randomly chosen from the other n natives different from A and B, thus at most n more colors are required. In total we need

$$\left\lfloor \frac{n^2}{4} \right\rfloor + 1 + n = \left\lfloor \frac{n^2}{4} + n + 1 \right\rfloor = \left\lfloor \frac{(n+2)^2}{4} \right\rfloor$$

colors, which proves our initial statement.

The **Romanian Mathematical Society** (Societatea de Ştiinţe Matematice din România) is a centenial non-for-profit organization of Romanian mathematicians. It is the founder, more than hundred years ago of the Romanian Mathematical Olympiad. The monthly journal "Gazeta Matematică" is worldwide known as well as other mathematical texts published by the Society during years.

`http://www.ssmro.com`

The **Theta Foundation** is a non-for-profit organization promoting mathematical knowledge.

The Theta Foundation is the editor of the "Journal of Operator Theory", a worldwide known mathematical periodical, and of some important texts in mathematical research. A series of translations into Romanian of some fundamental monographs in mathematics is also one of the Theta achievements.

`http://www.theta.ro`

ISBN 973–0–03138–X

The **Romanian Mathematical Society** (Societatea de Ştiinţe Matematice din România) is a centenial non-for-profit organization of Romanian mathematicians. It is the founder, more than hundred years ago of the Romanian Mathematical Olympiad. The monthly journal "Gazeta Matematică" is worldwide known as well as other mathematical texts published by the Society during years.

`http://www.ssmro.com`

The **Theta Foundation** is a non-for-profit organization promoting mathematical knowledge.
The Theta Foundation is the editor of the "Journal of Operator Theory", a worldwide known mathematical periodical, and of some important texts in mathematical research. A series of translations into Romanian of some fundamental monographs in mathematics is also one of the Theta achievements.

`http://www.theta.ro`

ISBN 973–0–03138–X