Mircea Becheanu
Mihai Bălună
Bogdan Enescu
Radu Gologan
Dinu Şerbănescu

R.M.C.

2002

ROMANIAN MATHEMATICAL COMPETITIONS

Societatea de Ştiinţe Matematice din România
and
Theta Foundation

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EDITORS:

DR. MIRCEA BECHEANU – **Main Editor**
Department of Mathematics, University of Bucharest
E-mail: bechmir@rms.ynibuc.ro)

DR. RADU GOLOGAN
Institute of Mathematics and
University "Politehnica", Bucharest
E-mail: rgologan@theta.ro

MIHAI BĂLUNĂ
National College "Mihai Viteazul", Bucharest

BOGDAN ENESCU
National College "B.P. Haşdeu, Buzău
E-mail: enescu@buzau.ro

DINU ŞERBĂNESCU
National College "Sf. Sava"
Bucharest

Distributed by:
Societatea de Ştiinţe Matematice din Romania
Str. Academiei 14, 70103 Bucureşti, Romania
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PREFACE

This book is a collection of the most beautiful problems proposed for the mathematical competitions during the academical year 2001-2002 in Romania. The participants to these competitions are junior-high and high-school students, hence all problems have elementary mathematical level, corresponding to the school curricula. Nevertheless, the authors try to introduce new ideas and techniques in order to help students to discover the beauty of mathematics.

The main mathematical competition in Romania is the National Mathematical Olympiad. It is held yearly since 1949 and a great number of young students take part every year since. It is organized in several rounds starting with a school examination for selection and ending with the Final National Contest. The Committee of the National Olympiad tries to select only original problems, proposed by Romanian mathematicians. Then, the list is completed by the committee, in order to insure a consistent examination. In this way, a list of problems was created every year, including the presented one.

Many other competitions of regional character are also organized by the Romanian Mathematical Society through its Departments. Some of these competitions have already reached a strong tradition being organized in honour of important Romanian Mathematicians: Gheorghe Ţiţeica, Gheorghe Vrânceanu, Gheorghe Mihoc, Grigore Moisil, Spiru Haret and others. Many interesting mathematical problems arise from these competitions too.

The authors of this book collected the most beautiful problems given at these competitions held in Fall 2001 and Spring 2002 with the aim to support the creation of elementary mathematical problems. These problems have an important contribution to the mathematical education of the young people. Our purpose was also to encourage mathematical teachers to be involved in finding of new problems.

The series Romanian Mathematical Competitions was created and supported by the Romanian Mathematical Society since 1994. In order to get a more exhaustive collection, its contend was extended this year.

We thank the Romanian Mathematical Society and the Theta Foundation for supporting the publishing of the present book.

Special thanks are due to Mrs. Luminiţa Stafi and Barbara Ionescu, from the Theta Foundation, who carefully typed most of the manuscript and contributed in time to the editing of this book.

Bucharest,
July 16^{th}, 2002

Mircea Becheanu
Radu Gologan

CONTENTS

Part I: THE 53$^{\text{th}}$ NATIONAL MATHEMATICAL OLYMPIAD

PROPOSED PROBLEMS

I.1. FIRST ROUND. CITY OF BUCHAREST

January 26, 2002

9$^{\text{th}}$ GRADE

PROBLEM 1. Find all positive integers a, b for which the number

$$\frac{\sqrt{2}+\sqrt{a}}{\sqrt{3}+\sqrt{b}}$$

is a rational number.

I.V. Maftei

PROBLEM 2. Let $ABCD$ be a unit square and M, N be interior points on the sides AB, BC respectively, such that

$$\frac{AM}{MB} = 7 \text{ and } \frac{CN}{NB} = 2.$$

Let P be the intersection point of the lines CM and DN.

a) Show that $13\overrightarrow{AP} = 12\overrightarrow{AB} + 5\overrightarrow{AD}$.

b) Compute the length AP.

Laurenţiu Panaitopol

PROBLEM 3. Find all real functions $f, g, h : \mathbf{R} \to \mathbf{R}$, such that

$$(x-y)f(x) + h(x) - xy + y^2 \leqslant h(y) \leqslant (x-y)g(x) + h(x) - xy + y^2,$$

for all real numbers x, y.

Marcel Chiriţă

PROBLEM 4. Let $ABCD$ be a rhombus and M, N, P be interior points on the sides AB, BC, CD respectively. Show that the centroid of the triangle MNP belongs to the line AC if and only if $AM + DP = BN$.

Marian Andronache

10$^{\text{th}}$ GRADE

PROBLEM 1. Solve in the set $\mathbf{C}$ of complex numbers the following equations:

a) $|z-a| + |z-b| = b-a$, where a, b are real numbers.

b) $|z| + |z-1| + |z-2| + |z-3| = 4$.

Petruş Alexandrescu and Sorin Rădulescu

PROBLEM 2. a) Let a be a real number, $a > 1$, and $f, g, h : \mathbf{R} \to \mathbf{R}$ be real functions such that $f(x) + g(x) + h(x) \geqslant 0$, for all $x \in \mathbf{R}$. Show that the equation

$$a^{f(x)} + a^{g(x)} + a^{h(x)} = 3$$

has solutions if and only if the functions f, g, h have common zeros.

b) Solve the equation

$$5^{1+\cos \pi x} + 2^{x^2-1} + 4^{1-|x|} = 3.$$

Valentin Matrosenco

PROBLEM 3. Let ABC be a triangle and M, N be the midpoints of the sides BC, AC respectively. It is known that the orthocenter of the triangle ABC and the centroid of the triangle AMN coincide. Find the angles of the triangle ABC.

Marian Andronache

PROBLEM 4. Let $(a_n)_{n\geqslant 1}$ be an arithmetic progression that contains the numbers 1 and $\sqrt{2}$. Show that any three numbers from the sequence $(a_n)_{n\geqslant 1}$ are not in a geometric progression.

Laurenţiu Panaitopol

11th GRADE

PROBLEM 1. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 4$. Show that the determinant

$$\begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix}$$

is not greater than 16.

Marcel Chiriţă

PROBLEM 2. Let $(x_n)_{n\geqslant 1}$ be an arithmetic progression of positive numbers. For any positive integer n, denote by $a(n)$ the arithmetic mean and by $g(n)$ the geometric mean of the first n terms of the progression. Compute

$$\lim_{n\to\infty} \frac{a(2n) - a(n)}{g(2n) - g(n)}.$$

Marcel Chiriţă

PROBLEM 3. Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers such that $a_n \geqslant -1$ and $a_n - a_{n+1} > a_n a_{n+1}$, for all $n \geqslant 1$. Show that $a_n > 0$ for all $n \geqslant 1$.

Dinu Şerbănescu

PROBLEM 4. Let A be a 2×2 matrix with entries in $\mathbf{C}$. For any positive integer n, denote by $x_n = \det(A^n + I)$. Show that if $x_1 = x_2 = 1$, then x_n is either 1 or 4.

Laurenţiu Panaitopol

12th GRADE

PROBLEM 1. Compute the following integrals:

a) $\displaystyle\int_{-1}^{1} \left(\frac{\sqrt{x^2+1} + x - 1}{\sqrt{x^2+1} + x + 1} \right) \mathrm{d}x$

b) $\displaystyle\int_{-1}^{1} \frac{\mathrm{d}x}{x^2 + x + 1 + \sqrt{x^4 + 3x^2 + 1}}.$

Marcel Chiriţă

PROBLEM 2. Let $\varphi : M \times M \to M$, $\varphi(x, y) = xy$, be a composition law possesing an identity element and satisfying the condition

• for any $a, b, x, y \in M$, such that $ab = xy$, it follows that $ax = by$.

Show that M is an abelian group with respect to φ.

Marian Andronache

PROBLEM 3. Let $f : \mathbf{R} \to \mathbf{R}$ be a function which has a derivative f' and let F be an antiderivative function of f. We assume that the following properties hold:

(i) the limit $\lim\limits_{x\to\infty} xf'(x)$ exists;

(ii) $\lim\limits_{x\to\infty} \frac{F(x)}{x} = 1$.

Compute the limits: $\lim\limits_{x\to\infty} xf'(x)$ and $\lim\limits_{x\to\infty} f(x)$.

Mihai Bălună

PROBLEM 4. We are given a finite group with n elements. Suppose the group contains two elements of order p, q, respectively, such that $p \geqslant 2$, $q \geqslant 2$, p, q are relatively prime and $p + q \geqslant n - 1$. Find n.

Laurenţiu Panaitopol

I.2. SECOND ROUND (DISTRICT LEVEL)

February 16, 2002

7th GRADE

PROBLEM 1. Find the number of representations of the number 180 in the form $180 = x + y + z$, where x, y, z are positive integers that are proportional with some three consecutive positive integers.

PROBLEM 2. A group of 67 students pass an examination consisting of six questions, labeled with the numbers 1 to 6. A correct answer to question n is quoted n points and for an incorect answer to the same question a student loses n points.

a) Find the least possible positive difference between any two final scores.

b) Show that at least four participants have the same final score.
c) Show that at least two students gave identical answers to all six questions.

Dan Brânzei and Mircea Fianu

PROBLEM 3. Let ABC be an equilateral triangle, G be its centroid and M be an interior point. Let O be the midpoint of the segment MG. Through the point M three segments are drawn that are parallel to the sides of the triangle and have their endpoints on the sides of the triangle.
a) Show that the point O is at equal distance to the midpoints of these three segments.
b) Show that the midpoints of the three segments are the vertices of an equilateral triangle.

M. Asiminoaie

PROBLEM 4. Let $ABCD$ be a rectangle and E, F be points on the segments BC and DC respectively, such that $\angle DAF = \angle FAE$. Show that if $DE + BE = AE$, then $ABCD$ is a square.

Mircea Fianu

8[th] GRADE

PROBLEM 1. Let x, y, z be positive real numbers such that $xyz(x+y+z) = 1$. Show that the following equality holds:

$$\sqrt{\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{z^2}\right)\left(z^2+\frac{1}{x^2}\right)} = (x+y)(y+z)(z+x).$$

Find some numbers x, y, z which satisfy the given property.

PROBLEM 2. a) Let x be a real number such that x^2+x and x^3+2x are rational numbers. Show that x is a rational number.
b) Show that there exist irrational numbers x such that x^2+x and x^3-2x are rational.

Florica Banu

PROBLEM 3. We are given a regular quadrilateral pyramid $VABCD$ and let O be the center of the square $ABCD$. The angle between two lateral opposite sides of the pyramid is $45°$. Denote by M the projection of the point A on the line CV, by N the symmetrical point of M with respect to the plane (VBD), and by P the symmetrical point of N with respect to O.
a) Show that the polyhedron $MDNBP$ is a regular pyramid.
b) Find the angle between the line ND and the plane (ABC).

Mircea Fianu

PROBLEM 4. Let $ABCDA'B'C'D'$ be a cube. On sides $AB, CC', D'A'$ one considers the points K, L, M respectively.
a) Show that $\sqrt{3}KL \geqslant KB + BC + CL$.
b) Show that $KL + LM + MK > 2\sqrt{3}AB$.

Dan Brânzei and Radu Gologan

9[th] GRADE

PROBLEM 1. Prove that for every real number x, the following inequality holds

$$\left[\frac{x+3}{6}\right] - \left[\frac{x+4}{6}\right] + \left[\frac{x+5}{6}\right] = \left[\frac{x+1}{2}\right] - \left[\frac{x+1}{3}\right].$$

Cristinel Mortici

PROBLEM 2. Let $ABCD$ be a cyclic quadrilateral and M be a point on its circumcircle. Let H_1, H_2, H_3, H_4 be the orthocenters of the triangles MAB, MBC, MCD, MDA respectively. Prove that:
a) $H_1H_2H_3H_4$ is a parallelogram.
b) $H_1H_3 = 2EF$.

Nicolae Mușuroia

PROBLEM 2. Let ABC be a triangle, G be its centroid and M, N, P be points on the sides AB, BC, CA respectively, such that

$$\frac{AM}{MB} = \frac{BN}{NC} = \frac{CP}{PA}.$$

Denote by G_1, G_2, G_3 the centroids of the triangles AMP, BMN, CNP respectively. Prove that:
a) the triangles ABC and $G_1G_2G_3$ have the same centroid;
b) for every point D in the plane (ABC), one has

$$3DG < DG_1 + DG_2 + DG_3 < DA + DB + DC.$$

Vasile Cornea and Dan Marinescu

PROBLEM 4. Let n be a positive integer, $n \geqslant 2$. Prove that:
a) if $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2$, then $a_1 + a_2 + \cdots + a_n \leqslant n$;
b) if x is a real number such that $1 \leqslant x \leqslant n$, then there are nonnegative real numbers $a_1, a_2, \ldots, a_n$, such that

$$x = a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2.$$

Romeo Ilie

10[th] GRADE

PROBLEM 1. Find a closed formula for x_n, $n \geqslant 2$, where $x_1 = 1$ and

$$4(x_1x_n + 2x_2x_{n-1} + 3x_3x_{n-2} + \cdots + nx_nx_1) = (n+1)(x_1x_2 + x_2x_3 + \cdots + x_nx_{n+1}),$$

for each $n \geqslant 1$.

Nicolae Papacu

PROBLEM 2. Solve in complex numbers the system:

$$x(x-y)(x-z)=3$$
$$y(y-x)(y-z)=3$$
$$z(z-x)(z-y)=3.$$

Mihai Piticari

PROBLEM 3. Let a, b be real numbers such that $3^a + 13^b = 17^a$ and $5^a + 7^b = 11^b$. Prove that $a < b$.

Cristinel Mortici

PROBLEM 4. For every positive integer n, $n \geqslant 2$, denote by $f(n)$ the minimal number of elements of a set S which satisfies the conditions:

(i) $1 \in S$ and $n \in S$;

(ii) every element of S, except 1, is a sum of two, possible not distinct, elements of S.

Prove that:

a) $f(n) \geqslant [\log_2 n] + 1$.

b) $f(n) = f(n+1)$ for infinitely many numbers n.

Dorel Miheţ

11th GRADE

PROBLEM 1. a) Let a and b be positive real numbers. Compute the limit

$$\lim_{n\to\infty} \sqrt{a+\sqrt{a+\cdots+\sqrt{a+\sqrt{b}}}},$$

the number of radicals being n.

b) Let $(a_n)_{n\geqslant 1}$ be a sequence of positive numbers and $(x_n)_{n\geqslant 1}$ be the sequence defined by

$$x_n = \sqrt{a_n + \sqrt{a_{n-1} + \cdots + \sqrt{a_2 + \sqrt{a_1}}}}.$$

Prove that:

(i) the sequence $(x_n)_{n\geqslant 1}$ is bounded if and only if the sequence $(a_n)_{n\geqslant 1}$ is bounded.

(ii) the sequence $(x_n)_{n\geqslant 1}$ is convergent if and only if the sequence $(a_n)_{n\geqslant 1}$ is convergent.

Valentin Matrosenco and Radu Gologan

PROBLEM 2. In a rectangular system of coordinates of a plane, we consider the points $A_n(n, n^3)$, where n runs over all positive integers and the point $B(0,1)$. Prove that:

a) for every integers $k \geqslant j > i \geqslant 1$ the points A_i, A_j, A_k are not on a line;

b) for every positive integers $1 \leqslant i_1 < i_2 < \cdots < i_{n-1} < i_n$, the following inequality holds:

$$\angle A_{i_1}OB + \angle A_{i_2}OB + \cdots + \angle A_{i_n}OB < \frac{\pi}{2}.$$

PROBLEM 3. a) Find a 3×3 matrix A with complex entries, $A \in M_3(\mathbf{C})$, such that $A^2 \neq 0$ and $A^3 = 0$.

b) Let n, p numbers which are 2 or 3. We assume that there exists a function $f : M_n(\mathbf{C}) \to M_p(\mathbf{C})$ with properties:

- f is a bijective function;
- $f(XY) = f(X) \cdot f(Y)$, for every $X, Y \in M_n(\mathbf{C})$.

Prove that $n = p$.

Ion Savu

PROBLEM 4. Let $f : \mathbf{R} \to \mathbf{R}$ be a function which satisfies the conditions:

(i) f has lateral limits in any point $a \in \mathbf{R}$ and

$$f(a-0) \leqslant f(a) \leqslant f(a+0);$$

(ii) for any real numbers a, b, $a < b$, one has

$$f(a-0) < f(b-0).$$

Prove that f is a monotonic increasing function.

Mihai Piticari and Sorin Rădulescu

12th GRADE

PROBLEM 1. Let A be a ring, $a \in A$, and n, k be integers such that $n \geqslant 2$, $k \geqslant 2$, $\underbrace{1+1+\ldots+1}_{n \text{ times}} = 0$ and $a^k = a + 1$. Prove that:

a) for every positive integer s, there exist non-negative integers $p_0, p_1, \ldots, p_{k-1}$ such that

$$a^s = p_0 \cdot 1 + p_1 \cdot a + \cdots + p_{k-1} \cdot a^{k-1};$$

b) there exists a pozitive integer m such that $a^m = 1$.

Marian Andronache

PROBLEM 2. a) For any positive integer n, let A_n be the ring

$$A_n = \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2 = \mathbf{Z}_2^n$$

Show that if $n \neq m$, then the rings A_n and A_m are not isomorphic but there exists a ring homomorphism $f : A_n \to A_m$.

b) Prove that there exist rings $B_1, B_2, \ldots, B_n, \ldots$ such that, no homomorphism exists between B_n and B_m, when $n \neq m$.

Barbu Berceanu

PROBLEM 3. For any real number a, $0 < a \leqslant 1$, we denote

$$I_n(a) = \int_0^a \ln(1 + x + \cdots + x^{n-1})\,\mathrm{d}x, \quad n \geqslant 2.$$

Compute the limit: $\lim_{n\to\infty} I_n(a)$.

Mihai Piticari

PROBLEM 4. Let $f : \mathbf{R} \to [0,\infty)$ be a continuous function which is periodic of period 1. Prove that:

a) $\displaystyle\int_a^{a+1} f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x,\ \forall\, a \in \mathbf{R}$;

b) $\displaystyle\lim_{n\to\infty}\int_0^1 f(x)f(nx)\,\mathrm{d}x = \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$.

Cristinel Mortici

I.3. FINAL ROUND

Râmnicu Vâlcea – March 18, 2002

7th GRADE

PROBLEM 1. Eight card players are seated around a table. One remarks that at some moment, any player and his two neighbours have alltogether an odd number of winning cards.

Show that any player has at that moment at least one winning card.

PROBLEM 2. Prove that any real number x, $0 < x < 1$, can be written as difference of two positive and less than 1 irrational numbers.

PROBLEM 3. Let $ABCD$ be a trapezoid and AB, respectively CD be its parallel edges. Find, with proof, the set of interior points P of the trapezoid which have the following property:

"P belongs to at least two lines each intersecting the segments AB and CD and each dividing the trapezoid in two other trapezoids with equal areas".

PROBLEM 4. a) An equilateral triangle of sides a is given and a triangle MNP is constructed under the following conditions: $P \in (AB)$, $M \in (BC)$, $N \in (AC)$, such that $MP \perp AB$, $NM \perp BC$ and $PN \perp AC$. Find the lenght of the segment MP.

b) Show that for any acute triangle ABC one can find points $P \in (AB)$, $M \in (BC)$, $N \in (AC)$, such that $MP \perp AB$, $NM \perp BC$ and $PN \perp AC$.

Mircea Fianu

8th GRADE

PROBLEM 1. For any number $n \in \mathbf{N}$, $n \geqslant 2$, denote by $P(n)$ the number of pairs (a, b) whose elements are positive integers such that

$$\frac{n}{a} \in (0,1), \quad \frac{a}{b} \in (1,2) \quad \text{and} \quad \frac{b}{n} \in (2,3).$$

a) Calculate $P(3)$.
b) Find n such that $P(n) = 2002$.

Mircea Fianu

PROBLEM 2. Given real numbers a, c, d, show that there exists at most one function $f : \mathbf{R} \to \mathbf{R}$ which satisfies:

$$f(ax + c) + d \leqslant x \leqslant f(x + d) + c, \text{ for any } x \in \mathbf{R}.$$

Laurenţiu Panaitopol

PROBLEM 3. Let $[ABCA'B'C']$ be a frustum of a regular pyramid. Let G and G' be the centroids of bases ABC and $A'B'C'$ respectively. It is known that $AB = 36$, $A'B' = 12$ and $GG' = 35$.

a) Prove that the planes (ABC'), (BCA'), (CAB') have a common point P, and the planes $(A'B'C)$, $(B'C'A)$, $(C'A'B)$ have a common point P', both situated on GG'.

b) Find the length of the segment $[PP']$.

Dan Brânzei

PROBLEM 4. The right prism $[A_1A_2A_3\cdots A_nA_1'A_2'\cdots A_n']$, $n \in \mathbf{N}$, $n \geqslant 3$, has a convex polygon as its base. It is known that $A_1A_2' \perp A_2A_3'$, $A_2A_3' \perp A_3A_4', \ldots, A_{n-1}A_n' \perp A_nA_1'$, $A_nA_1' \perp A_1A_2'$. Show that:
a) $n = 3$;
b) the prism is regular.

Mircea Fianu

9th GRADE

PROBLEM 1. Let a, b, c be positive numbers such that $ab+bc+ca = 1$. Show that:

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geqslant \sqrt{3} + \frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}.$$

Dinu Teodorescu

PROBLEM 2. Let ABC be a right triangle $\angle A = 90°$ and let $M \in (AB)$ such that $\frac{AM}{MB} = 3\sqrt{3} - 4$. It is known that the symmetric point of M with respect to the line GI lies on AC. Find the measure of angle B (*G is the centroid and I is the center of the incircle*).

Marian Andronache

PROBLEM 3. Let k and n be positive integers, $n > 2$. Show that the equation:

$$x^n - y^n = 2^k$$

has no positive integer solutions.

Romeo Ilie

PROBLEM 4. Find all functions $f : \mathbf{N} \to \mathbf{N}$ which satisfy the equality

$$f(3x + 2y) = f(x)f(y),$$

for all nonnegative integers x, y.

Gheorghe Iurea

10th GRADE

PROBLEM 1. Let X, Y, Z, T be four points in the plane. The segments $[XY]$ and $[ZT]$ are said to be *connected*, if there is some point O in the plane such that the triangles OXY and OZT are rightangled in O and isosceles.

Let $ABCDEF$ be a convex hexagon such that the pairs of segments $[AB]$, $[CE]$, and $[BD]$, $[EF]$ are *connected*. Show that the points A, C, D and F are the vertices of a parallelogram and that the segments $[BC]$ and $[AE]$ are *connected*.

Bogdan Enescu

PROBLEM 2. Find all real polynomials f, g which satisfy the condition:

$$(x^2 + x + 1) \cdot f(x^2 - x + 1) = (x^2 - x + 1) \cdot g(x^2 + x + 1),$$

for all $x \in \mathbf{R}$.

Marcel Chiriţă

PROBLEM 3. Find all real numbers a, b, c, d, e in the interval $[-2, 2]$, that satisfy:

$$a + b + c + d + e = 0$$
$$a^3 + b^3 + c^3 + d^3 + e^3 = 0$$
$$a^5 + b^5 + c^5 + d^5 + e^5 = 10.$$

Titu Andreescu

PROBLEM 4. Let $I \subseteq \mathbf{R}$ be an interval and $f : I \to \mathbf{R}$ be a function such that:

$$|f(x) - f(y)| \leqslant |x - y|, \text{ for all } x, y \in I.$$

Show that f is monotonic on I if and only if, for any $x, y \in I$, either $f(x) \leqslant f\left(\frac{x+y}{2}\right) \leqslant f(y)$ or $f(y) \leqslant f\left(\frac{x+y}{2}\right) \leqslant f(x)$.

Romeo Ilie

11th GRADE

PROBLEM 1. In the Carthesian plane xOy one considers the hyperbola

$$\Gamma = \left\{ M(x, y) \in \mathbf{R}^2 \,\middle|\, \frac{x^2}{4} - y^2 = 1 \right\}$$

and a conic Γ', disjoint from Γ. Let $n(\Gamma, \Gamma')$ be the maximal number of pairs of points $(A, A') \in \Gamma \times \Gamma'$ such that $AA' \leqslant BB'$, for any $(B, B') \in \Gamma \times \Gamma'$.

For each $p \in \{0, 1, 2, 4\}$, find the equation of Γ' such that $n(\Gamma, \Gamma') = p$. Justify the answer.

(The following curves are considered here as conics: *the circle, the ellipse, the hyperbola and the parabola.*)

Barbu Berceanu

PROBLEM 2. Let $f : \mathbf{R} \to \mathbf{R}$ be a function which has limits at any point and has no local extreme. Show that:

a) f is continuous;

b) f is strictly increasing or strictly decreasing.

Mihai Piticari and Sorin Rădulescu

PROBLEM 3. Let $A \in M_4(\mathbf{C})$ be a non-zero matrix.

a) If $\operatorname{rank}(A) = r < 4$, prove that one can find two invertible matrices $U, V \in M_4(\mathbf{C})$, such that:

$$UAV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the r-unit matrix.

b) Show that if A and A^2 have the same rank k, then the matrix A^n has rank k, for any $n \geqslant 3$.

Marian Andronache and Ion Savu

PROBLEM 4. Let $f : [0, 1] \to [0, 1]$ be a continuous and bijective function. Describe the set:

$$A = \{f(x) - f(y) \,|\, x, y \in [0, 1] \setminus \mathbf{Q}\}.$$

(Consider the following result to be known: *there is no one-to-one function between the set of irrational numbers and* $\mathbf{Q}$.)

Radu Gologan

12th GRADE

PROBLEM 1. Let A be a ring.

a) Show that the set $Z(A) = \{a \in A \,|\, ax = xa, \text{ for all } x \in A\}$ is a subring of the ring A.

b) Prove that, if any commutative subring of A is a field, then A is a field.

(A subset $B \subset A$ is called a *subring* if the following are true: $x, y \in B$ implies $xy, x - y \in B$ and $1 \in B$, 1 being the unit of A.)

Ion Savu

PROBLEM 2. Let $f : [0,1] \to \mathbf{R}$ be an integrable function such that:

$$0 < \left| \int_0^1 f(x)\mathrm{d}x \right| \leqslant 1.$$

Show that there exist $x_1 \neq x_2$, $x_1, x_2 \in [0,1]$, such that:

$$\int_{x_1}^{x_2} f(x)\mathrm{d}x = (x_1 - x_2)^{2002}.$$

Radu Gologan

PROBLEM 3. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous and bounded function such that:

$$x \int_x^{x+1} f(t)\mathrm{d}t = \int_0^x f(t)\mathrm{d}t, \text{ for any } x \in \mathbf{R}.$$

Prove that f is a constant function.

Mihai Piticari

PROBLEM 4. Let K be a field having $q = p^n$ elements, where p is a prime number and $n \geqslant 2$ is an arbitrary integer number. For each $a \in K$, one defines the polynomial $f_a = X^q - X + a$. Show that:

a) $f = (X^p - X)^q - (X^p - X)$ is divisible by f_1;

b) f_a has at least p^{n-1} essentialy different irreducible factors $K[X]$.

(One may use the following classical result: *any finite field is commutative.*)

Marian Andronache

I.4. ELEMENTARY SCHOOL OLYMPIAD

City of Bucharest

May 5, 2002

5th GRADE

PROBLEM 1. Show that the number

$$\frac{36}{5\cdot 7} - \frac{1}{5\cdot 6\cdot 7} - \frac{1}{6\cdot 7\cdot 8} - \frac{1}{6\cdot 8}$$

is an integer.

PROBLEM 2. We consider the number

$$N = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \cdots + \frac{11}{10^{11}}.$$

Show that $0.12345679 < N < 0.1234568$.

PROBLEM 3. A sports contest was organized during four days. The medals were distributed as follows: each day half of the existing medals were awarded and one more. How many medals were awarded each of the four days?

PROBLEM 4. The sets A and B consist each of a finite number of consecutive positive integers. Let a be the arithmetic mean of the elements in A and b be the arithmetic mean of elements in B. The arithmetic mean of a and b is 12 and it is known that $A \cap B = \{12\}$. Find the maximal number of elements in the set $A \cup B$.

6th GRADE

PROBLEM 1. Let $A = \{a \in \mathbf{Z} \mid -2000 \leqslant a \leqslant 2000\}$.

a) Find the sum of elements of A.

b) Show that the sum of absolute values of elements of A is a perfect square.

PROBLEM 2. Find positive integers a, b which satisfy the conditions:

(i) $6a + b = 330$;

(ii) the least common multiple of a and b is 12 times greater than the greatest common divisor of a and b.

PROBLEM 3. Let a, b, c be positive integers such that

$$\frac{a+b}{bc} = \frac{b+c}{ca} = \frac{c+a}{ab}.$$

Show that $a = b = c$.

PROBLEM 4. Let ABC be an isosceles triangle. The base of the triangle ABC is AC, the length of AC is a and $\angle B = 70°$. On the segments AB, AC are given the points D, E respectively, such that $DA + AE = a$. On the segments AC, BC are given the points F, G respectively, such that $FC + CG = a$. The points E, F are distinct. Find the angle between the lines DF and EG.

I.5. SHORTLISTED PROBLEMS
for the National Mathematical Olympiad
Final Round

PROBLEM 1. Let a, b, c be decimal digits and n a positive integer such that

$$1 + 2 + \cdots + n - \overline{abcabc}.$$

Find the number $\overline{abc}$.

7th Grade, Gheorghe Moraru

PROBLEM 2. Let $ABCD$ be a trapezoid with A a right angle and parallel sides CD, CD. Prove that $AB^2 + CD^2 = BC^2 + AD^2$ if and only if $AC \perp BD$.

7^{th} Grade, Dorin Popa

PROBLEM 3. A sheet of paper is a square with sides 10 centimeters. Prove that one can cut an equilateral triangle of sides 10.3 cm but one cannot cut an equilateral triangle of sides 10.4 cm.

7^{th} Grade, Laurenţiu Panaitopol

PROBLEM 4. Let x be a real number such that $x^{2002} = x^{2003} + 1$. Then x cannot be rational.

7^{th} Grade, P. Simion and S. Smarandache

PROBLEM 5. In a right triangle the sides are a, b, c in standard notations. Prove that

$$1 < \frac{a}{b+c} + \frac{bc}{a^2} \leqslant \frac{1+\sqrt{2}}{2}.$$

7^{th} Grade, Petre Stângescu

PROBLEM 6. Let a, b, c be real numbers such that $x = a^2 + b^2 + c^2$ is positive. Prove that

$$a^3 + b^3 + c^3 - 3abc \leqslant x\sqrt{x}.$$

8^{th} Grade, Marcel Chiriţă

PROBLEM 7. Given positive numbers a, b, c such that $abc = 1$, show that

$$(a+b)(b+c)(c+a) \leqslant \left(\frac{a+b+c}{2}\right)^6.$$

8^{th} Grade, Valer Pop

PROBLEM 8. The numbers a, b, c are given, such that $a, b, c > 1$ and $abc = 2\sqrt{2}$. Prove that

$$(a+1)(b+1)(c+1) > 8(a-1)(b-1)(c-1).$$

8^{th} Grade, Gheorghe Molea

PROBLEM 9. A convex n-gone is given and a is a positive integer such that the number of diagonals equals $\frac{n}{a}$. Find n.

8^{th} Grade, Valer Pop

PROBLEM 10. We are given a right parallelipiped of diagonal 1 and M an arbitrary point inside it. Denote by $s(M)$ the sum of squares of distances of M to the eight vertices of the parallelipiped. Find the maximal and minimal value of $s(M)$.

8^{th} Grade, Valentin Matrosenco

PROBLEM 11. Given positive integers m and n prove that one can find positive integers a and b, such that

$$(m^4 - m^2 + 1)(n^4 - n^2 + 1) = a^2 + b^2.$$

8^{th} Grade, Bogdan Enescu

PROBLEM 12. Given A and B, subsets of the set of real numbers, we denote

$$AB = \{xy \mid x \in A,\, y \in B\}.$$

Given a real number a, find the finite subsets $X \subset \mathbf{R}$ such that $XX \subseteq \{a\}X$.

9^{th} Grade, Marcel Ţena

PROBLEM 13. We are given positive numbers a, b, c such that $a + b + c = 1$. Prove that

$$5(a^2 + b^2 + c^2) \leqslant 6(a^3 + b^3 + c^3) + 1.$$

9^{th} Grade, Mihai Piticari and Dan Popescu

PROBLEM 14. Let A, B, C, D be distinct points on a circle of center O. Prove that if there are nonzero real numbers x, y such that

$$|x\overrightarrow{OA} + y\overrightarrow{OB}| = |x\overrightarrow{OB} + y\overrightarrow{OC}| = |x\overrightarrow{OC} + y\overrightarrow{OD}| = |x\overrightarrow{OD} + y\overrightarrow{OA}|,$$

then $ABCD$ is a square.

9^{th} Grade, Manuela Prajea

PROBLEM 15. Given positive numbers a, b, c such that $a + b + c = 1$, prove that

$$(ab)^{\frac{5}{4}} + (bc)^{\frac{5}{4}} + (ca)^{\frac{5}{4}} < \frac{1}{4}.$$

9^{th} Grade, Dinu Teodorescu

PROBLEM 16. Let n be an integer, $n \geqslant 3$. Find all complex numbers z such that both z^n and $(1+z)^n$ are real numbers.

10^{th} Grade, Gheorghe Iurea

PROBLEM 17. Given the sequence of positive numbers $(a_n)_{n \geqslant 1}$ that satisfy $a_{n+1} = \sqrt{6 - 2a_n^2}$, for any $n \geqslant 1$, prove that it is constant.

10^{th} Grade, Laurenţiu Panaitopol

PROBLEM 18. We are given complex numbers a, b, c, mutually distinct, such that $|a| + |b| + |c| = 1$ and $|a-b|^2 + |b-c|^2 + |c-a|^2 > 8$. Prove that

$$|(a+b)(b+c)(c+a)| \leqslant 1.$$

10^{th} Grade, Dan Nedeianu

PROBLEM 19. Let A, B be complex matrices of size 3×3. Prove that

$$\det(AB - BA) = \operatorname{tr}(AB(AB - BA)BA).$$

11^{th} Grade, Radu Gologan

PROBLEM 19. Given A and B complex matrices of size $n \times n$, such that B is obtained from A by some permutations of its rows, show that either $\det(A+B) = 0$ or for some positive integer r we have $\det(A+B) = 2^r \det(A)$.

11^{th} Grade, Cornel Berceanu

PROBLEM 20. Let $(x_n)_{n\geqslant 1}$ be a sequence of positive numbers that satisfies $(x_{n+1}-x_n)(x_nx_{n+1}-1)\leqslant 0$ and $\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n}=1$. Prove that $(x_n)_{n\geqslant 1}$ has a finite limit.

11^{th} Grade, Mihai Piticari

PROBLEM 21. Find with proof, all continuous periodical functions, $f:\mathbf{R}\to\mathbf{R}$, such that for all real x and any integer n we have $|nf(x)f(nx)|\leqslant 1$.

11^{th} Grade, Cristinel Mortici

PROBLEM 22. Given a continuous non-constant function $f:[0,1]\to[0,1]$, prove that there exist $x_1,x_2\in[0,1]$, $x_1\neq x_2$, such that

$$|f(x_1)-f(x_2)|=|x_1-x_2|^2.$$

11^{th} Grade, Radu Gologan

PROBLEM 23. Given a real number a, find all continuous functions $f:\mathbf{R}\to\mathbf{R}$, such that $f(f(x))+f(x)=2x+a$, for any real x.

11^{th} Grade, Marcel Chiriţă

PROBLEM 24. We consider the following sets of 2×2 complex matrices: M^2 is the set of all matrices that have square roots and M^3 is the set of matrices that have cubic roots. Prove that $M^2=M^3$.

11^{th} Grade, Mihai Bălună

PROBLEM 25. Let A be a ring without divisors of 0 and such that there is $a\in A$ satisfying $a^2-a=1+1$. Prove that $1+1+1=0$ (here 1 stands for the ring unit and 0 is the zero element in A).

12^{th} Grade, Tiberiu Agnola

PROBLEM 26. Let A be a complex matrix of size 2×2 which is neither 0 or I. Given an integer n, consider the set of matrices $S_n=\{X\,|\,X^n=A\}$. Prove that the following conditions are equivalent:

a) S_n is a multiplicative group isomorphic with the group of n-roots of unity;

b) $A^2=A$.

12^{th} Grade, Marian Andronache

PROBLEM 27. In a ring A we have $1\neq 0$. Suppose that there is an integer n and an element $x\in A$ such that $x^{6n+2}=x$. Prove that $1-x+x^2$ is an invertible element in A.

12^{th} Grade, Nicolae Papacu

PROBLEM 28. Let $f:[0,1]\to\mathbf{R}$ be an infinitely derivable function such that $f^{(n)}(0)=0$ for $n\geqslant 2002$ and

$$\lim_{n\to\infty}\int_0^1|f^{(n)}(x)|\mathrm{d}x=0.$$

Prove that f is polynomial.

12^{th} Grade, Daniel Jinga and Ionel Popescu

PROBLEM 29. Let K and L be fields and $\mathcal{F}$ be the group of all functions $f:K\to L$. A "(K,L)-integral" will be a group morphism $I:\mathcal{F}\to L$, which has the following properties:

(i) for any $a\in K$, $f\in\mathcal{F}$ we have $I(f\circ t_a)=I(f)$, where $t_a(x)=x+a$;

(ii) $I(1)=1_L$, unde $1(x)=1_L$.

Prove that:

a) if $p\geqslant 3$ is a prime number, then $I(f)=f(\widehat{0})+f(\widehat{1})+\cdots+f(\widehat{p-1})$ is a $(\mathbf{Z}_p,\mathbf{Z}_2)$ integral;

b) if a "(K,L)-integral" exists, then there are no morphisms between the fields K and L.

12^{th} Grade, Barbu Berceanu

Part II: THE 53th NATIONAL MATHEMATICAL OLYMPIAD

SOLUTIONS

II.1. FIRST ROUND. CITY OF BUCHAREST

9th GRADE

PROBLEM 1. Find all positive integers a, b for which the number

$$\frac{\sqrt{2}+\sqrt{a}}{\sqrt{3}+\sqrt{b}}$$

is a rational number.

SOLUTION. Let $\alpha \in \mathbf{Q}$ such that

$$\frac{\sqrt{2}+\sqrt{a}}{\sqrt{3}+\sqrt{b}} = \alpha.$$

We write the equality under the form

$$\sqrt{a} - \alpha\sqrt{b} = \alpha\sqrt{3} - \sqrt{2}.$$

After squaring, we obtain

$$a + \alpha^2 b - 2\alpha\sqrt{ab} = 3\alpha^2 + 2 - 2\alpha\sqrt{6}.$$

It follows that

$$\sqrt{ab} - \sqrt{6} = \beta \in \mathbf{Q}.$$

By squaring the equality $\sqrt{ab} = \sqrt{6} + \beta$, we obtain $ab = 6 + \beta^2 + 2\beta\sqrt{6}$, which is the same as $2\beta\sqrt{6} = ab - 6 - \beta^2$. The last equality is possible only when $\beta = 0$; that is $ab = 6$. Taking into account all possible cases, one obtains:

$$a = 1,\ b = 6, \text{ that is } \alpha = \frac{\sqrt{2}+1}{\sqrt{3}+\sqrt{6}} = \frac{1}{\sqrt{6}}, \text{ an irrational number}$$

$$a = 2,\ b = 3, \text{ that is } \alpha = \frac{2\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}, \text{ an irrational number}$$

$$a = 3,\ b = 2, \text{ that is } \alpha = \frac{\sqrt{2}+\sqrt{3}}{\sqrt{3}+\sqrt{2}} = 1 \in \mathbf{Q}$$

$$a = 6,\ b = 1, \text{ that is } \alpha = \frac{\sqrt{2}+\sqrt{6}}{\sqrt{3}+1} = \sqrt{2}, \text{ an irrational number}$$

Thus the answer is $a = 3$, $b = 2$.

PROBLEM 2. Let $ABCD$ be a unit square and M, N be interior points on the sides AB, BC respectively, such that

$$\frac{AM}{MB} = 7 \text{ and } \frac{CN}{NB} = 2.$$

Let P be the intersection point of the lines CM and DN.

a) Show that $13\overrightarrow{AP} = 12\overrightarrow{AB} + 5\overrightarrow{AD}$.

b) Compute the length AP.

SOLUTION. We shall use coordinates. Take the origin of a rectangular system of coordinates to be $A(0,0)$, such that one has $B(1,0), C(1,1), M(\frac{7}{8},0)$ and $N(1,\frac{1}{3})$. The lines DN and CM have respectively the equations:

$$(CM)\ 8x - y = 7$$
$$(DN)\ 2x + 3y = 3.$$

Solving the system we obtain for the coordinates of P: $x = \frac{12}{13}$, $y = \frac{5}{13}$. Since $\overrightarrow{AB} = e_1$, $\overrightarrow{AD} = e_2$ are the orthogonal unit vectors of the choosen coordinate system, we have: $\overrightarrow{AP} = \frac{12}{13}e_1 + \frac{5}{13}e_2 = \frac{12}{13}\overrightarrow{AB} + \frac{5}{13}\overrightarrow{AD}$.

It follows, by standard computation that

$$AP = \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{1}{13}\right)^2} = 1,$$

answering thus to the last question.

PROBLEM 3. Find all real functions $f, g, h : \mathbf{R} \to \mathbf{R}$, such that

$$(x-y)f(x) + h(x) - xy + y^2 \leqslant h(y) \leqslant (x-y)g(x) + h(x) - xy + y^2,$$

for all real numbers x, y.

SOLUTION. Comparing the extreme parts of the given inequality, one obtains

$$(x-y)f(x) + h(x) - xy + y^2 \leqslant (x-y)g(x) + h(x) - xy + y^2,$$

that is $(x-y)f(x) \leqslant (x-y)g(x)$ for all $x, y \in \mathbf{R}$. For $y = x+1$ this gives $f(x) \leqslant g(x)$ and for $y = x-1$ it reduces to $f(x) \geqslant g(x)$. Hence $f(x) = g(x)$ for all real x. The given inequality then becomes

$$(1) \qquad (x-y)f(x) + h(x) - xy + y^2 = h(y), \text{ for all } x, y \in \mathbf{R}.$$

For $x = 0$ in (1), we obtain

$$(2) \qquad h(y) = y^2 - f(0)y + h(0), \text{ for all } y \in \mathbf{R}.$$

It follows that h is a quadratic function. Denote $f(0) = a$, $h(0) = b$ and use formula (2) in (1), to obtain

$$(x-y)f(x) + x^2 - ax + b - xy + y^2 = y^2 - ay + b, \text{ for all } x, y \in \mathbf{R},$$

that is $(x-y)f(x) + x(x-y) - (x-y)a = 0$, for all $x, y \in \mathbf{R}$. Since x, y are arbitrary, the last equality gives $f(x) + x - a = 0$ for all $x \in \mathbf{R}$. We may conclude that

$$f(x) = g(x) = -x + a$$
$$h(x) = x^2 - ax + b,$$

for all $x \in \mathbf{R}$, a, b being real constants.

PROBLEM 4. Let $ABCD$ a rhombus and M, N, p be interior points on the sides AB, BC, CD respectively. Show that the centroid of the triangle MNP belongs to the line AC if and only if $AM + DP = BN$.

SOLUTION. Let O be the intersection point of the diagonals of the given rhombus. We shall use vectors. Put

$$\frac{AM}{AB} = m, \quad \frac{BN}{BC} = n, \quad \frac{DP}{DC} = p.$$

Since $AB = BC = DC$, the condition $AM + DP = BN$ is equivalent to $m + p = n$. Consider the vectors

$$\overrightarrow{OM} = (1-m)\overrightarrow{OA} + m\overrightarrow{OB}, \ \overrightarrow{ON} = (1-n)\overrightarrow{OB} + n\overrightarrow{OC}, \ \overrightarrow{OP} = (1-p)\overrightarrow{OD} + p\overrightarrow{OC}.$$

Let G be the centroid of the triangle MNP. Since $\overrightarrow{OA} = -\overrightarrow{OC}, \overrightarrow{OD} = -\overrightarrow{OB}$ and $3\overrightarrow{OG} = \overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OP}$, we obtain

$$3\overrightarrow{OG} = (m + n + p - 1)\overrightarrow{OC} + (m - n + p)\overrightarrow{OB}.$$

The point G belongs to the line AC if and only if the vectors $\overrightarrow{OG}$ and $\overrightarrow{OC}$ are colinear, that is if and only if $m - n + p = 0$.

10th GRADE

PROBLEM 1. Solve in the set $\mathbf{C}$ of complex numbers the following equations:
a) $|z-a| + |z-b| = b - a$, where a, b are real numbers;
b) $|z| + |z-1| + |z-2| + |z-3| = 4$.

SOLUTION. a) The geometric interpretation of the distance in the complex plane and the triangle's inequality, show that z should be a point on the segment $[a, b]$.

b) By previous arguments:
• $|z| + |z-3| \geqslant |z - z + 3| = 3$, and equality holds if and only if z is real and $0 \leqslant z \leqslant 3$;
• $|z-1| + |z-2| \geqslant |z - 1 - z + 2| = 1$, and equality holds if and only if z is real and $1 \leqslant z \leqslant 2$;

By adding these inequalities, we obtain

$$|z| + |z-1| + |z-2| + |z-3| = 4,$$

if and only if z is real and $1 \leqslant z \leqslant 2$.

PROBLEM 2. a) Let a be a real number, $a > 1$, and $f, g, h : \mathbf{R} \to \mathbf{R}$ be real functions such that $f(x) + g(x) + h(x) \geqslant 0$, for all $x \in \mathbf{R}$. Show that the equation

$$a^{f(x)} + a^{g(x)} + a^{h(x)} = 3$$

has solutions if and only if the functions f, g, h have common zeros.

b) Solve the equation

$$5^{1+\cos \pi x} + 2^{x^2-1} + 4^{1-|x|} = 3.$$

SOLUTION. a) By the AM-GM inequality, we have

$$1 = \frac{1}{3}\left(a^{f(x)} + a^{g(x)} + a^{h(x)}\right) \geqslant \sqrt[3]{a^{f(x)}a^{g(x)}a^{h(x)}} = \sqrt[3]{a^{f(x)+g(x)+h(x)}} \geqslant \sqrt[3]{a^0} = 1.$$

Therefore, if x is a solution of the given equation, we must have

$$a^{f(x)} = a^{g(x)} = a^{h(x)} \text{ and } f(x) + g(x) + h(x) = 0.$$

That is $f(x) = g(x) = h(x)$ and consequently $f(x) = g(x) = h(x) = 0$.

b) The equation may be written in the form

$$2^{(1+\cos \pi x)\log_2 5} + 2^{x^2-1} + 2^{2(1-|x|)} = 3.$$

Hence, considering $f(x) = (1 + \cos \pi x)\log_2 5$, $g(x) = x^2 - 1$, $h(x) = 2(1 - |x|)$, we have $f(x) + g(x) + h(x) = (1 + \cos \pi x)\log_2 5 + (|x| - 1)^2 \geqslant 0$. The previous result implies that the solution of the equation consists of the common roots of the functions f, g, h. These are $x = 1$ and $x = -1$.

PROBLEM 3. Let ABC be a triangle and M, N be the midpoints of the sides BC, AC respectively. It is known that the orthocenter of the triangle ABC and the centroid of the triangle AMB coincide. Find the angles of the triangle ABC.

SOLUTION. We shall use complex numbers. Take the circumcenter O of the triangle ABC as the origin of the coordinate plane. Denote by a, b, c the complex numbers that are the afixes of the points A, B, C respectively. The orthocenter of the triangle ABC corresponds to the complex number $h = a + b + c$. The centroid of the triangle AMN corresponds to the complex number

$$g = \frac{1}{3}\left(a + \frac{b+c}{2} + \frac{c+a}{2}\right) = \frac{3a + b + 2c}{6}.$$

From $g = h$ we get $3a + 5b + 4c = 0$. Without loss of generality, we may assume $a = 1$ and consequently $|b| = |c| = 1$. The previous equality becomes $3 + 5b + 4c = 0$ and taking conjugates one also has $3 + 5\bar{b} + 4\bar{c} = 0$, or $3 + \frac{5}{b} + \frac{4}{c} = 0$. Solving for b and c the system given by the two equalities, one obtains, either $c = \mathrm{i}$, $b = -\frac{3}{5} - \frac{4}{5}\mathrm{i}$ or $c = -\mathrm{i}$, $b = -\frac{3}{5} + \frac{4}{5}\mathrm{i}$.

The obtained triangles are congruent since they are symmetrical with respect to the real axis. By standard computation we obtain $\angle B = \frac{\pi}{4}$, $\angle A = \arctan 3$ and $\angle C = \arctan 2$.

PROBLEM 4. Let $(a_n)_{n \geqslant 1}$ be an arithmetic progression that contains the numbers 1 and $\sqrt{2}$. Show that any three numbers from the sequence $(a_n)_{n \geqslant 1}$ are not in a geometric progression.

SOLUTION. Assume that the arithmetic progression $a_1, a_2, \ldots a_n, \ldots$, of ratio r satisfies for some positive integers k, l, $a_k = 1$ and $a_l = \sqrt{2}$. Then $\sqrt{2} - 1 = (l-k)r$ or $r = \frac{\sqrt{2}-1}{l-k}$. Proceeding by contradiction, assume that a_m, a_n and a_p are in a geometric progression, in that order. Denote for convenience $\frac{m-k}{l-k} = N$, $\frac{n-k}{l-k} = M$ and $\frac{p-k}{l-k} = P$. We have

$$a_m = a_k + (m-k)r = 1 + M(\sqrt{2} - 1)$$
$$a_n = a_k + (n-k)r = 1 + N(\sqrt{2} - 1)$$
$$a_p = a_k + (l-k)r = 1 + P(\sqrt{2} - 1).$$

The condition $a_n^2 = a_m a_p$ then reads

$$\left((1-N)+N\sqrt{2}\right)^2 = \left((1-M)+M\sqrt{2}\right)\left((1-P)+P\sqrt{2}\right),$$

that is

$$(1-N)^2 + 2N^2 + 2N(1-N)\sqrt{2} = ((1-M)(1-P)+2MP) + (M(1-P)+P(1-M))\sqrt{2}.$$

Since M, N, P are rationals, the following equalities must hold

$$3N^2 - 2N = 3MP - M - P$$
$$2N - 2N^2 = M + P - 2MP.$$

Summing up, we conclude $N^2 = MP$ which gives $2N = M+P$ and $(M+P)^2 = 4MP$. Thus $(M-P)^2 = 0$ which gives $M = N = P$ and consequently $m = n = p$, a contradiction.

11th GRADE

PROBLEM 1. Let a, b, c be real numbers such that $a^2+b^2+c^2 = 4$. Show that the determinant

$$\begin{vmatrix} a+b & b+c & c+a \\ c+a & a+b & b+c \\ b+c & c+a & a+b \end{vmatrix}$$

is not greater than 16.

SOLUTION. The given determinant is circulant, and its value is

$$D = 2(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

Denote $s = a+b+c$. The hypothesis implies $s^2 = 4 + 2(ab+bc+ca)$, that is $ab+bc+ca = \frac{s^2-4}{2}$. It follows $D = 2s\left(4 - \frac{s^2-4}{2}\right) = s(12-s^2)$. The condition $D \leqslant 16$ can be written as follows: $s(12-s^2) \leqslant 16$ or $(s-2)^2(s+4) \geqslant 0$. It remains to prove that $s+4 \geqslant 0$. From the well-known inequality

$$(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$$

we obtain $s^2 \leqslant 12$. Therefore $s \geqslant -2\sqrt{3} > -4$.

PROBLEM 2. Let $(x_n)_{n\geqslant 1}$ be an arithmetic progression of positive numbers. For any positive integer n, denote by $a(n)$ the arithmetic mean and by $g(n)$ the geometric mean of the first n terms of the progression. Compute

$$\lim_{n\to\infty} \frac{a(2n)-a(n)}{g(2n)-g(n)}.$$

SOLUTION. Let r be the ratio of the given progression $(x_n)_{n\geqslant 1}$. By standard computations, we obtain

$$a(n) = \frac{x_1+x_n}{2} = \frac{2x_1+(n-1)r}{2}$$

and

$$\lim_{n\to\infty} \frac{a(n)}{n} = \lim_{n\to\infty} \frac{2x_1+(n-1)r}{2n} = \frac{r}{2}.$$

Consider the sequence given by $y_n = \frac{n^n}{x_1x_2\cdots x_n}$ $(y_n > 0)$. Since

$$\frac{y_{n+1}}{y_n} = \left(\frac{n+1}{n}\right)^n \cdot \frac{n+1}{x_{n+1}} = \left(1+\frac{1}{n}\right)^n \frac{n+1}{x_1+nr},$$

we derive

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{\mathrm{e}}{r}.$$

Using the Cauchy-D'Alembert convergence criterion, we conclude

$$\lim_{n\to\infty} \frac{n}{g(n)} = \lim_{n\to\infty} \sqrt[n]{y_n} = \frac{\mathrm{e}}{r}.$$

Being subsequences of $\left(\frac{a(n)}{n}\right)_{n\geqslant 1}$ and $\left(\frac{g(n)}{n}\right)_{n\geqslant 1}$, the sequences $\left(\frac{a(2n)}{2n}\right)_{n\geqslant 1}$ and $\left(\frac{g(2n)}{2n}\right)_{n\geqslant 1}$ are convergent to the same limits, respectively. Therefore, we obtain

$$\lim_{n\to\infty} \frac{a(2n)-a(n)}{g(2n)-g(n)} = \lim_{n\to\infty} \frac{\frac{a(2n)}{2n} - \frac{1}{2}\cdot\frac{a(n)}{n}}{\frac{g(2n)}{2n} - \frac{1}{2}\cdot\frac{g(n)}{n}} = \frac{\frac{r}{2} - \frac{r}{4}}{\frac{r}{\mathrm{e}} - \frac{1}{2}\cdot\frac{r}{\mathrm{e}}} = \frac{\mathrm{e}}{2}.$$

PROBLEM 3. Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers such that $a_n \geqslant -1$ and $a_n - a_{n+1} > a_n a_{n+1}$ for all $n \geqslant 1$. Show that $a_n > 0$ for all $n \geqslant 1$.

SOLUTION. Since $a_n^2 \geqslant 0 > a_n a_{n+1} + a_{n+1} - a_n$, we get

(1) $$a_n(a_n+1) > a_{n+1}(a_n+1).$$

We can improve $a_n > -1$ and from (1) we deduce $a_n > a_{n+1}$. Thus the sequence is decreasing and bounded. We conclude that it is convergent. Let $l = \lim_{n\to\infty} a_n$. Taking limits in the original inequality one obtains $l = 0$. Since $(a_n)_{n\geqslant 1}$ is decreasing we obtain $a_n > 0$ for any n.

ALTERNATIVE SOLUTION. From $a_n > a_{n+1}(1+a_n)$ and $1+a_n > 0$ we obtain $\frac{a_n}{1+a_n} > a_{n+1}$. The following inequality is true for all numbers x, $x > -1$:

$$x \geqslant \frac{x}{1+x}.$$

Therefore

(2) $$a_n \geqslant \frac{a_n}{1+a_n} > a_{n+1}.$$

It follows that the sequence is convergent to l and (2) implies $l \geqslant \frac{l}{1+l} \geqslant l$ that is $l = 0$. By consequence $a_n > l = 0$.

PROBLEM 4. Let A be a 2×2 matrix with entries in $\mathbf{C}$. For any positive integer n, denote by $x_n = \det(A^n + I)$. Show that if $x_1 = x_2 = 1$, then x_n is either 1 or 4.

SOLUTION. Let $P(X) = \det(A - XI)$ be the characteristic polynomial of the matrix A. It is known that $P(X) = X^2 - aX + b$ where $a = \operatorname{tr} A$ and $b = \det(A)$. Moreover, A verifies the characteristic equation, that is $A^2 - aA + bI = 0$.

Using the hypothesis, we have $x_1 = \det(A+I) = P(-1) = 1$, that is $a+b = 0$.

In order to use $x_2 = 1$, observe that $A^2+I = (A+\mathrm{i}I)(A-\mathrm{i}I)$, where $\mathrm{i}^2 = -1$. Then we have

$$1 = x_2 = P(-\mathrm{i})P(\mathrm{i}) = (-1 + a\mathrm{i} + b)(-1 - a\mathrm{i} + b) = (b-1)^2 + a^2.$$

Consequently $a = -b$ and $a^2 + b^2 - 2b = 0$. One obtains either $b = 0$ or $b = 1$ and $a = 0$ or $a = -1$. In case $a = b = 0$ it follows $A^2 = 0$ and $A^n = 0$ for all $n \geqslant 2$, and thus $x_n = 1$ for all n.

Assume $a = -1, b = 1$. Then $P(X) = X^2 + X + 1$, implying $A^2 + A + I = 0$ and also $A^3 = I$ $(0 = (A - I)(A^2 + A + I) = A^3 - I)$. By induction $A^{3k} = I$, and $x_{3k} = \det(2I) = 4$. In the same way $A^{3k+1} = A^{3k}A = A$, $x_{3k+1} = x_1 = 1$ and $A^{3k+2} = A^{3k}A^2 = A^2$, $x_{3k+2} = x_2 = 1$. It follows that $x_n \in \{1, 4\}$ for any n.

12th GRADE

PROBLEM 1. Compute the following integrals:

a) $\displaystyle\int_{-1}^{1} \left(\frac{\sqrt{x^2+1}+x-1}{\sqrt{x^2+1}+x+1}\right) \mathrm{d}x$

b) $\displaystyle\int_{-1}^{1} \frac{\mathrm{d}x}{x^2+x+1+\sqrt{x^4+3x^2+1}}$.

SOLUTION. a) Let

$$f(x) = \frac{x^2+1-(x-1)^2}{(\sqrt{x^2+1}+x+1)(\sqrt{x^2+1}-x+1)} = \frac{x}{\sqrt{x^2+1}+1}.$$

It is obvious that f is an odd function. It follows that f^3 is also odd, and

$$\int_{-1}^{1} f^3(x)\mathrm{d}x = 0.$$

b) Consider $g : [-1, 1] \to \mathbf{R}$ defined by $g(x) = \dfrac{1}{x^2 - x + \sqrt{x^4 + 3x^2 + 1}}$. We have the equality

$$g(x) = \frac{x^2+x+1-\sqrt{x^4+3x^2+1}}{(x^2+x+1)-(x^4+3x^2+1)} = \frac{x^2+x+1-\sqrt{x^4+3x^2+1}}{2(x^3+x)},$$

which is valid for all nonzero x. Because $g(x) + g(-x) = \frac{1}{x^2+1}$ for $x \neq 0$, and $g(0) = \frac{1}{2}$, we can write

$$\int_{-1}^{1} g(x)\mathrm{d}x = \int_{-1}^{0} g(x)\mathrm{d}x + \int_{0}^{1} g(x)\mathrm{d}x = -\int_{1}^{0} g(-y)\mathrm{d}x + \int_{0}^{1} g(x)\mathrm{d}x$$
$$= \int_{1}^{0} (g(x) + g(-x))\,\mathrm{d}x = \int_{0}^{1} \frac{1}{x^2+1}\mathrm{d}x = \arctan x\,\Big|_0^1 = \frac{\pi}{4}.$$

PROBLEM 2. Let $\varphi : M \times M \to M$, $\varphi(x, y) = xy$, be a composition law possesing an identity element and satisfying the condition

- for any $a, b, x, y \in M$, such that $ab = xy$, it follows that $ax = by$.

Show that M is an abelian group with respect to φ.

SOLUTION. We prove that the composition law is associative. Let e, $e \in M$, be the identity element. For any $a, b \in M$ we have $ab = e(ab)$ which implies $a = ae = b(ab) = (ab)b$. Since $(ab)a = b$ and $c(bc) = b$, for all $a, b, c \in M$, we deduce $(ab)a = c(bc)$, that is $(ab)c = a(bc)$, proving thus the associativity.

From $ae = ea = e$ we obtain $aa = ee = e$, that is $a^2 = e$, which in turn proves that any element $a \in M$ is invertible.

To prove that M is abelian, observe that from the hypothesis $ab = xy$ implies $ax = by$ and $by = ax$ implies $ba = yx$. Therefore $ab = xy$ implies $ba = yx$. Let $ab = c$. From $ab = ce$ we thus conclude $ba = ec = c = ba$. Since $a, b \in M$ are arbitrary, we obtain the commutativity.

PROBLEM 3. Let $f : \mathbf{R} \to \mathbf{R}$ be a function which has a derivative f' and let F be a antiderivative function of f. We assume that the following properties hold:

(i) the limit $\lim\limits_{x\to\infty} xf'(x)$ exists;

(ii) $\lim\limits_{x\to\infty} \frac{F(x)}{x} = 1$.

Compute the limits: $\lim\limits_{x\to\infty} xf'(x)$ and $\lim\limits_{x\to\infty} f(x)$.

SOLUTION. Let $l = \lim\limits_{x\to\infty} xf'(x)$, where l may be $\pm\infty$. As $(xf(x) - F(x))' = xf'(x)$, we can use L'Hospital rule to compute

$$\lim_{x\to\infty} \frac{xf(x) - F(x)}{x} = l.$$

Since

$$f(x) = \frac{xf(x) - F(x)}{x} + \frac{F(x)}{x},$$

we get $\lim\limits_{x\to\infty} f(x) = l + 1$. As $f(x) = \frac{F'(x)}{(x)'}$, using again L'Hospital rule we find $\lim\limits_{x\to\infty} \frac{F(x)}{x} = l + 1$. We conclude $l = 0$ and $\lim\limits_{x\to\infty} f(x) = 1$.

PROBLEM 4. We are given a finite group with n elements. Suppose that the group contains two elements of order p, q, respectively, such that $p \geqslant 2$, $q \geqslant 2$, p, q are relatively prime and $p + q \geqslant n - 1$. Find n.

SOLUTION. Let G be the given group and $|G| = n$. Since $(p, q) = 1$, there exists an element of order pq in G. Hence $pq|n$ and also $p, q \leqslant n \leqslant p + q + 1$. It follows that $\frac{p+q+1}{pq} = \frac{1}{p} + \frac{1}{q} + \frac{1}{pq} \geqslant 1$.

On the other hand

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{pq} \leqslant \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1.$$

It follows that there is only one possibility, that is $p = 2$, $q = 3$ and $n = 6$.

II.2. SECOND ROUND

7th GRADE

PROBLEM 1. Find the number of representations of the number 180 in the form $180 = x + y + z$, where x, y, z are positive integers that are proportional with some three consecutive positive integers.

SOLUTION. Let $n \geqslant 2$ such that $\frac{x}{n-1} = \frac{y}{n} = \frac{z}{n+1}$. We have more: $\frac{x}{n-1} = \frac{y}{n} = \frac{z}{n+1} = \frac{x+y+z}{3n} = \frac{60}{n}$. It follows that $y = 60$, $x = \frac{n-1}{n} \cdot 60$ and $z = \frac{n+1}{n} \cdot 60$. Since $(n-1, n) = (n, n+1) = 1$ it follows that n should be a divisor of 60. That is n takes one of the values $n = 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$. For any of these numbers we get a represention. So, the required number is 11.

PROBLEM 2. A group of 67 students pass an examination consisting of six questions, labeled with the numbers 1 to 3. A correct answer to question n is quoted n points and for an incorect answer to the same question a student loses n points.

a) Find the least possible positive difference between any two final scores.

b) Show that at least four participants have the same final score.

c) Show that at least two students gave identical answers to all six questions.

SOLUTION. a) The least positive difference between two answers to the same question is 2. It is obtained when two students gave different answers to the question 1 and the same answers to remaining questions.

b) The greatest score is $1 + 2 + 3 + 4 + 5 + 6 = 21$ and the least score is -21. The difference between two scores is an even number. So, all scores should be elements of the set $S = \{-21, -19, -17, \ldots, -1, 1, \ldots, 17, 19, 21\}$. The set S has 22 elements. If every score has been obtained by at most three students, we can have no more than 66 students. So, at least four students have the same score.

c) Every final score is a sum of the form:

$$\pm 1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6.$$

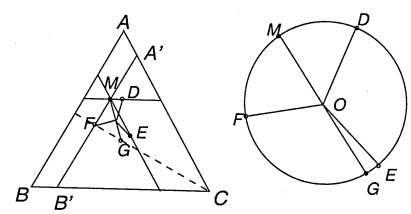
There are $2^6 = 64$ possibilities to choose signes + or −, so there are 64 possibilities to give answers. Since there are 67 students it follows that at least two students gave identical answers.

PROBLEM 3. Let ABC be an equilateral triangle, G be its centroid and M be an interior point. Let O be the midpoint of the segment MG. Through the point M three segments are drawn that are parallel to the sides of the triangle and have their endpoints on the sides of the triangle.

a) Show that the point O is at equal distance to the midpoints of these three segments.

b) Show that the midpoints of the three segments are the vertices of an equilateral triangle.

SOLUTION. a) Let D, E, F be the midpoints of the segments parallel to BC, CA, AB. The segment CF is perpendicular to the segment MF, so it passes through G. It follows that the triangle GFM is rectangular in F, MG is its hypothenuse and then $OF = \frac{1}{2}MG$. In the same way one obtains $OD = OE = \frac{1}{2}MG$ (see figure).



b) We have the points D, E, F, G, M on a circle with center O (see Figure 2). Assume that M is an interior point of the angle $\angle FGD$; other situations are similar. We have: $\angle FOD = \angle FOM + \angle MOD = 360° - 2\angle FMG - 2\angle GMD = 360° - 2\angle FMD = 360° - 240° = 120°$. Also, $\angle FOE = \angle FOG + \angle GOE = 2\angle OMF + 2\angle OME = 2\angle FME = 120°$. It follows that $\angle EOD = 120°$. Therefore, the points D, E, F cut the circle in three equal arcs and so, the triangle DEF is equilateral.

PROBLEM 4. Let $ABCD$ be a rectangle and E, F be points on the segments BC and DC respectively, such that $\angle DAF = \angle FAE$. Show that if $DE + BE = AE$, then $ABCD$ is a square.

SOLUTION. Let AF intersects BC in M and the perpendicular in A to AM intersects the line BC in N. Since $\angle DAF = \angle MAE$ it follows that ΔAEM is an isosceles triangle and $AE = EM$. Since ΔMAN is rectangular triangle in A it follows that ΔAEN is isosceles triangle with equal angles $\angle ENA = \angle EAN$. Hence, $NE = AE$. Because B is interior point of the segment NE, we have $DF + FB = AE = EB + BN$ and deduce $DF = BN$. We also have $\angle BAN = \angle DAF$. Therefore, the rectangle triangles DAF and BAN are congruent. We obtain $AB = AD$.

8th GRADE

PROBLEM 1. Let x, y, z be positive real numbers such that $xyz(x+y+z) = 1$. Show that the following equality holds:

$$\sqrt{\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{z^2}\right)\left(z^2+\frac{1}{x^2}\right)} = (x+y)(y+z)(z+x).$$

Find some numbers x, y, z which satisfy the given property.

SOLUTION. a) Using the condition we obtain:

$$x^2+\frac{1}{y^2} = x^2+\frac{xyz(x+y+z)}{y^2} = x^2+\frac{xz(x+y+z)}{y}$$
$$= \frac{x^2y+xz(x+y+z)}{y} = \frac{x(y+z)(x+z)}{y}.$$

In the same way we obtain the equalities:

$$y^2+\frac{1}{z^2} = \frac{y(z+x)(y+x)}{z} \quad \text{and} \quad z^2+\frac{1}{x^2} = \frac{z(x+y)(z+y)}{x}.$$

After multiplication of these equalities we obtain

$$\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{z^2}\right)\left(z^2+\frac{1}{x^2}\right) = (x+y)^2(y+z)^2(z+x)^2.$$

b) One may take $x = y = 1$ and z to be a solution of the equation $z(z+2) = 1$. Equivalently, $z^2+2z-1 = 0$. The positive solution is $z = \frac{\sqrt{5}-1}{2}$.

Another possibility is to take $x = y = z$ and obtain the solution of $3x^4 = 1$: $x = \frac{1}{\sqrt[4]{3}}$.

PROBLEM 2. a) Let x be a real number such that x^2+x and x^3+2x are rational numbers. Show that x is a rational number.

b) Show that there exist irrational numbers x such that x^2+x and x^3-2x are rational.

SOLUTION. a) Denote $x^2+x = a$, $x^3+2x = b$, where $a, b \in \mathbf{Q}$. Then

$$b = x^3+x^2-x^2-x+x+2x = x(x^2+x)-(x^2+x)+3x$$
$$= ax-a+3x = x(a+3)-a.$$

We shall prove that $a \neq -3$. Indeed, if $x^2+x = -3$ one obtains $x^2+x+3 = 0$ and $\left(x+\frac{1}{2}\right)^2+\frac{11}{4} = 0$, which contradicts that x is a real number. So, since $a \neq -3$ we obtain

$$x = \frac{a+b}{a+3},$$

which proves that x is a rational number.

b) Let $x^3+x = a$ and $x^3-2x = c$, where $a, c \in \mathbf{Q}$. Using a similar method we obtain

$$x(a-1) = a+c.$$

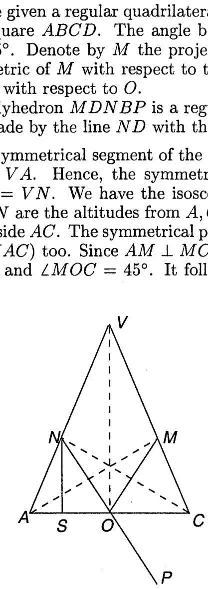
We may choose $a = 1$; other way we get $x \in \mathbf{Q}$. For $a = 1$, we obtain $x = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$, which is an irrational number. For both values of x, one obtains $c = x^3-2x = -a = -1$ and c is a rational number.

PROBLEM 3. We are given a regular quadrilateral pyramid $VABCD$ and let O be the center of the square $ABCD$. The angle between two lateral opposite sides of the pyramid is 45°. Denote by M the projection of the point A on the line CV, by N the symmetric of M with respect to the plane (VBD), and by P the symmetric point of N with respect to O.

a) Show that the polyhedron $MDNBP$ is a regular pyramid.

b) Find the angle made by the line ND with the plane (ABC).

SOLUTION. a) The symmetrical segment of the side VC with respect to the plane (VBD) is the side VA. Hence, the symmetric point of M is the point N of VA such that $VM = VN$. We have the isosceles triangle VAC in which $\angle AVC = 45°$ and AM, CN are the altitudes from A, C respectively. The point O is the median point of the side AC. The symmetrical point P of N with respet to O is situated in the plane (VAC) too. Since $AM \perp MC$ we have $MO = AO = OC$. Then $\triangle MOC$ is isosceles and $\angle MOC = 45°$. It follows that MO is orthogonal on NO.



Since O is the midpoint of the segments BD and NP, it follows that $BNDP$ is a parallelogram. Since BD is orthogonal on the plane (VAC), we obtain that BD is orthogonal on NP so $BNDP$ is a rhombus. Because $BD = AC = NP$, $BNDP$ is a square, since it is a rhombus with equal diagonals.

Also we have $BD \perp MO$, $MO \perp NO$, so MO is orthogonal on the plane $(BNDP)$. It follows that $MBNDP$ is a regular pyramid.

b) Let S be the projection of N on the side AC. Since $\angle NOS = 45°$ we obtain $NS\sqrt{2} = NO$. On the other hand, from $NO = OD$ and $NO \perp OD$ we obtain $ND = NO\sqrt{2}$, in $\triangle NOD$. So,

$$\sin \angle NDS = \frac{NS}{ND} = \frac{2NS}{2ND} = \frac{NO\sqrt{2}}{2NO\sqrt{2}} = \frac{1}{2}.$$

We obtain that $\angle NDS = 30°$.

PROBLEM 4. Let $ABCDA'B'C'D'$ be a cube on which sides $AB, CC', D'A'$ one considers the points K, L, M respectively.

a) Show that $\sqrt{3}KL \geqslant KB + BC + CL$.

b) Show that $KL + LM + MK > 2\sqrt{AB}$.

SOLUTION. a) + b) Let a be the lenght of the side of the cube. For simplicity let $KB = x$, $CL = y$, $A'M = z$, where $0 \leqslant x, y, z \leqslant a$. By using Pythagora's theorem we obtain:

$$KL^2 = x^2 + y^2 + a^2$$
$$KM^2 = (a-x)^2 + z^2 + a^2$$
$$ML^2 = (a-z)^2 + (a-y)^2 + a^2.$$

Using the well-known inequality:

$$(1) \qquad 3(a^2 + b^2 + c^2) \geqslant (a+b+c)^2$$

we obtain from each of above:

$$\sqrt{3}KL \geqslant x + y + a = KB + BC + CL$$
$$\sqrt{3}KM \geqslant (a-x) + z + a$$
$$\sqrt{3}ML \geqslant (a-z) + (a-y) + a.$$

By adding these inequalities one obtains

$$\sqrt{3}(KL + LM + ML) \geqslant 6a \Rightarrow KL + LM + ML \geqslant 2\sqrt{3}AB.$$

The equality occurs if and only if all three inequalities deduced from (1) are equalities. But in (1), we have equality if and only if $a = b = c$. In this case we should have $x = y = a$ and $a - x = z = a$, which is a contradiction. Therefore

$$KL + LM + MK > 2\sqrt{3}AB.$$

AUTHOR'S REMARK. It can be shown (for example by using the Cauchy-Buniakowsky inequality) that the minimum in b) is $\frac{3\sqrt{3}}{2}AB$.

9th GRADE

PROBLEM 1. Prove that for every real number x, the following equality holds

$$\left[\frac{x+3}{6}\right] - \left[\frac{x+4}{6}\right] + \left[\frac{x+5}{6}\right] = \left[\frac{x+1}{2}\right] - \left[\frac{x+1}{3}\right].$$

SOLUTION. Denote $\dfrac{x+1}{6} = y$. The equality can be written:

$$\left\lfloor y + \frac{1}{3}\right\rfloor - \left\lfloor y + \frac{1}{2}\right\rfloor + \left\lfloor y + \frac{2}{3}\right\rfloor = \lfloor 3y \rfloor - \lfloor 2y \rfloor.$$

It can be proved by using the identities of Hermite:

$$\lfloor 2y \rfloor = \lfloor y \rfloor + \left\lfloor y + \frac{1}{2}\right\rfloor$$
$$\lfloor 3y \rfloor = \lfloor y \rfloor + \left\lfloor y + \frac{1}{3}\right\rfloor + \left\lfloor y + \frac{2}{3}\right\rfloor.$$

Alternative solution. We can represent x under the form $x = 6k + y$, where $k \in \mathbf{Z}$ and $y \in [0, 6)$. The identity becomes:

$$\left\lfloor \frac{y+3}{6}\right\rfloor - \left\lfloor \frac{y+4}{6}\right\rfloor + \left\lfloor \frac{y+5}{6}\right\rfloor = \left\lfloor \frac{y+1}{2}\right\rfloor - \left\lfloor \frac{y+1}{3}\right\rfloor.$$

It can be easily checked by taking each of the cases:

$$y \in [0,1), \quad y \in [1,2), \quad y \in [2,3), \quad y \in [3,4), \quad [4,5), \quad y \in [5,6).$$

PROBLEM 2. Let $ABCD$ be a cyclic quadrilateral and M be a point on its circumcircle. Let H_1, H_2, H_3, H_4 be the orthocenters of the triangles MAB, MBC, MCD, MDA respectively. Prove that:

a) $H_1H_2H_3H_4$ is a paralleogram.

b) $H_1H_3 = 2EF$.

SOLUTION. a) We shall use vector algebra. The incenter of all triangles MAB, MBC, MCD, MDA is O. Hence, by Sylvester's formula we have:

$$\overrightarrow{OH_1} = \overrightarrow{OM} + \overrightarrow{OA} + \overrightarrow{OB}; \quad \overrightarrow{OH_2} = \overrightarrow{OM} + \overrightarrow{OB} + \overrightarrow{OC}; \quad \overrightarrow{OH_3} = \overrightarrow{OM} + \overrightarrow{OC} + \overrightarrow{OD}$$

and

$$\overrightarrow{OH_4} = \overrightarrow{OM} + \overrightarrow{OD} + \overrightarrow{OA}.$$

More computations give:

$$\overrightarrow{H_1H_2} = \overrightarrow{OH_2} - \overrightarrow{OH_1} = \overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{OH_3} - \overrightarrow{OH_4} = \overrightarrow{H_4H_3}.$$

b) Using again vectors:

$$\overrightarrow{H_1H_3} = \overrightarrow{OH_3} - \overrightarrow{OH_1} = \overrightarrow{OC} + \overrightarrow{OD} - \overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{AD} + \overrightarrow{BC} = 2\overrightarrow{EF}.$$

Hence $H_1H_3 = 2EF$.

PROBLEM 3. Let ABC be a triangle, G its centroid and M, N, P be points on the sides AB, BC, CA respectively, such that

$$\frac{AM}{MB} = \frac{BN}{NC} = \frac{CP}{PA}.$$

Denote by G_1, G_2, G_3 the centroids of the triangles AMP, BMN, CNP respectively. Prove that:

a) the triangles ABC and $G_1G_2G_3$ have the same centroid;

b) for every point D in the plane (ABC), one has

$$3DG < DG_1 + DG_2 + DG_3 < DA + DB + DC.$$

SOLUTION. a) Let $\dfrac{AM}{MB} = \dfrac{BN}{NC} = \dfrac{CP}{PA} = \lambda$. Then

$$\overrightarrow{GM} = \frac{1}{\lambda+1}\overrightarrow{GA} + \frac{\lambda}{\lambda+1}\overrightarrow{GB}, \quad \overrightarrow{GN} = \frac{1}{\lambda+1}\overrightarrow{GB} + \frac{\lambda}{\lambda+1}\overrightarrow{GC}$$

and
$$\overrightarrow{GP} = \frac{1}{\lambda+1}\overrightarrow{GC} + \frac{\lambda}{\lambda+1}\overrightarrow{GA}.$$

Therefore
$$\begin{aligned}\overrightarrow{GG_1} + \overrightarrow{GG_2} + \overrightarrow{GG_3} &= \frac{1}{3}(\overrightarrow{GA} + \overrightarrow{GM} + \overrightarrow{GN} + \overrightarrow{GB} + \overrightarrow{GM}\\ &\quad + \overrightarrow{GP} + \overrightarrow{GC} + \overrightarrow{GP} + \overrightarrow{GA})\\ &= \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 0.\end{aligned}$$

This proves that G is the centroid of $\triangle G_1G_2G_3$.

b) The first inequality follows from
$$3\overrightarrow{DG} = \overrightarrow{DG_1} + \overrightarrow{DG_2} + \overrightarrow{DG_3},$$

which is a consequence of the previous result. Then apply the length of vectors and the obvious fact that $\overrightarrow{DG_1}, \overrightarrow{DG_2}, \overrightarrow{DG_3}$ are not colinear vectors.

The second inequality comes from
$$\begin{aligned}|\overrightarrow{DG_1}| &= \left|\frac{2}{3}\overrightarrow{DA} + \frac{\lambda}{3(\lambda+1)}\overrightarrow{DB} + \frac{1}{3(\lambda+1)}\overrightarrow{DC}\right|\\ &< \frac{2}{3}DA + \frac{\lambda}{3(\lambda+1)}DB + \frac{1}{3(\lambda+1)}DC.\end{aligned}$$

and two other similar inequalities, which are added together.

PROBLEM 4. Let n be a positive integer, $n \geqslant 2$. Prove that:

a) if $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2$, then $a_1 + a_2 + \cdots + a_n \leqslant n$.

b) if x is a real number such that $1 \leqslant x \leqslant n$, then there are nonnegative real numbers $a_1, a_2, \ldots, a_n$, such that
$$x = a_1 + a_2 + \cdots + a_n = a_1^2 + a_2^2 + \cdots + a_n^2.$$

SOLUTION. a) By Cauchy-Schwarz inequality we have:
$$a_1 + \cdots + a_n \leqslant \sqrt{n} \cdot \sqrt{a_1^2 + \cdots + a_n^2}.$$

When $a_1 + \cdots + a_n = a_1^2 + \cdots + a_n^2$, the required inequality follows.

b) Assume $k \leqslant x \leqslant k+1$, where $k < n$. Then we can take $a_3 = \cdots = a_{n-k} = a_{n-k+1} = 0$, $a_{n-k+2} = \cdots = a_{n-1} = a_n = 1$ and it remains to find a_1, a_2 such that
$$a_1 + a_2 = a_1^2 + a_2^2 = x - (k-1).$$

Let observe that $1 \leqslant x - k + 1 \leqslant 2$. Denote $x - k + 1 = y$. Since $1 \leqslant y \leqslant 2$, the equation
$$a^2 - 2ya + y^2 - y = 0$$

has positive roots a_1, a_2, which are required numbers.

10^{th} GRADE

PROBLEM 1. Find a closed formula for x_n, $n \geqslant 2$, where $x_1 = 1$ and
$$4(x_1x_n + 2x_2x_{n-1} + 3x_3x_{n-2} + \cdots + nx_nx_1) = (n+1)(x_1x_2 + x_2x_3 + \cdots + x_nx_{n+1}),$$
for each $n \geqslant 1$.

SOLUTION. Put $n = 1$ in the recursive relation and obtain $4x_1^2 = 2x_1x_2$. Since $x_1 = 1$, it follows $x_2 = 2$.

For $n = 2$ in the same relation, we obtain
$$4(x_1x_2 + 2x_2x_1) = 3(x_1x_2 + x_2x_3).$$

Since $x_1 = 1$, $x_2 = 2$, it follows $x_3 = 3$.

We shall prove by inclusion that $x_n = n$ for all n. Assume that $x_k = k$ for all k, $1 \leqslant k \leqslant n$. Then we have:
$$\begin{aligned}&4\sum_{k=1}^{n} kx_kx_{n+1-k} = (n+1)\sum_{k=1}^{n} x_kx_{k+1}\\ \Rightarrow\quad &4\sum_{k=1}^{n} k^2(n+1-k) = (n+1)\sum_{k=1}^{n-1} k(k+1) + (n+1)nx_{n+1}\\ \Rightarrow\quad &4(n+1)\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k^3 = (n+1)\sum_{k=1}^{n-1} k^2 + (n+1)\sum_{k=1}^{n-1} k + n(n+1)x_{n+1}\\ \Rightarrow\quad &4\frac{n(n+1)^2(2n+1)}{6} - 4\frac{n^2(n+1)^2}{4}\\ &\quad = \frac{n(n+1)(n-1)(2n-1)}{6} + \frac{(n+1)n(n-1)}{2} + n(n+1)x_{n+1}\\ \Rightarrow\quad &x_{n+1} = \frac{4(n+1)(2n+1)}{6} - n(n+1) - \frac{(n-1)(2n-1)}{6} - \frac{n-1}{2}\\ &\quad = \frac{(n+1)(8n+4-6n)}{6} - \frac{2n^2-2}{6} = n+1.\end{aligned}$$

PROBLEM 2. Solve in complex numbers the system:
$$\begin{aligned}x(x-y)(x-z) &= 3\\ y(y-x)(y-z) &= 3\,.\\ z(z-x)(z-y) &= 3\end{aligned}$$

SOLUTION. In any solution (x, y, z) we have $x \neq 0$, $y \neq 0$, $z \neq 0$ and $x \neq y$, $y \neq z$, $z \neq x$. We can divide each equation by others and obtain new equations:
$$(1)\qquad \begin{aligned}x^2 + y^2 &= yz + zx\\ y^2 + z^2 &= xy + zx\\ z^2 + x^2 &= xy + yz.\end{aligned}$$

By adding them one obtains the equality:

$$x^2+y^2+z^2=xy+yz+zx. \tag{2}$$

After substracting equations (1), the second from the first, one obtains $x+y+z=0$. By squaring this identity one obtains an improvement of (2):

$$x^2+y^2+z^2=xy+yz+zx=0. \tag{3}$$

Using (3) in (1) one obtains:

$$x^2=zy, \quad y^2=zx, \quad z^2=xy \tag{4}$$

and also:

$$x^3=y^3=z^3=xyz.$$

It follows that x,y,z are distinct roots of the same complex number $a=xyz$. From $x^3=y^3=z^3=xyz=a$ we obtain

$$x=\sqrt[3]{a}, \quad y=\varepsilon\sqrt[3]{a}, \quad z=\varepsilon^2\sqrt[3]{a}, \tag{5}$$

where $\varepsilon^2+\varepsilon+1=0$, $\varepsilon^3=1$. When introduce relations (5) in the first equation of the original system, one obtains $a^3(1-\varepsilon)(1-\varepsilon^2)=3$. Taking into account the computation:

$$(1-\varepsilon)(1-\varepsilon^2)=1-\varepsilon-\varepsilon^2+1=3,$$

we have $a^3=1$. Hence, we obtain using (5) that (x,y,z) is a permutation of the set $\{1,\varepsilon,\varepsilon^2\}$.

PROBLEM 3. Let a,b be real numbers such that $3^a+13^b=17^a$ and $5^a+7^b=11^b$. Prove that $a<b$.

SOLUTION. Assume by contradiction that $a\geqslant b$. Then $13^a\geqslant 13^b$ and $5^a\geqslant 5^b$. From the equality $3^a+13^b=17^a$ one obtains $3^a+13^a\geqslant 17^a$. Equivalently, $\left(\frac{3}{17}\right)^a+\left(\frac{13}{17}\right)^a\geqslant 1$. The real function $f(x)=\left(\frac{3}{17}\right)^x+\left(\frac{13}{17}\right)^x$ is monotonic decreasing and $f(1)=\frac{3}{17}+\frac{13}{17}=\frac{14}{17}<1$. Since $f(a)\geqslant 1\geqslant f(1)$, we obtain $a<1$.

From the equality $5^a+7^b=11^b$ one obtains $5^b+7^b\leqslant 11^b$. Equivalently, $\left(\frac{5}{11}\right)^b+\left(\frac{7}{11}\right)^b\leqslant 1$. The real function $g(x)=\left(\frac{5}{11}\right)^x+\left(\frac{7}{11}\right)^x$ is monotonic decreasing and $g(1)=\frac{5}{11}+\frac{7}{11}=\frac{12}{11}>1$. Since $g(b)\leqslant 1<g(1)$, we obtain $b>1$.

The inequalities $a<1<b$ are in contradiction with the original supposition: $a\geqslant b$. Since we get a contradiction, it follows that we can only have $a<b$.

PROBLEM 4. For every positive integer n, $n\geqslant 2$, denote by $f(n)$ the minimal number of elements of a set S which satisfies the two conditions:

(i) $1\in S$ and $n\in S$;

(ii) every element of S, except 1, is a sum of two, possible not distinct, elements of S.

Prove that:

a) $f(n)\geqslant[\log_2 n]+1$.

b) $f(n)=f(n+1)$ for infinitely many numbers n.

SOLUTION. a) Let $n\geqslant 2$, $f(n)=k$ and S be a set of k elements $1=a_1<a_2<\cdots<a_k=n$ as required. Then $a_2=a_1+a_1=1+1=2$ and either $a_3=a_1+a_1$ or $a_3=a_1+a_2$. It follows that $a_3=2$ or $a_3=4$; in any case $a_3\leqslant 2^2$. We assume by induction that $a_i\leqslant 2^{i-1}$. Then $a_{i+1}=a_r+a_s\leqslant 2^{r-1}+2^{s-1}\leqslant 2^{i-1}+2^{i-1}=2^i$. It follows that $n=a_k\leqslant 2^{k-1}$ and $k\geqslant 1+\log_2 n\geqslant 1+[\log_2 n]$.

b) The previous argument shows that if S has $1+[\log_2 n]$ elements, then $n=2^{k-1}$. Therefore, it follows that $f(2^k+1)\geqslant k+2$ and $f(2^k+2)\geqslant k+2$ for all integers $k\geqslant 2$. One the other hand, the sets

$$S_1=\{1,2,2^2,\ldots,2^k,2^k+1\}, \quad S_2=\{1,2,2^2,\ldots,2^k,2^k+2\}$$

have $k+2$ elements and satisfy the requirements (i) and (ii). So, $f(2^k+1)=f(2^k+2)=k+2$.

11th GRADE

PROBLEM 1. a) Let a and b be positive real numbers. Compute the limit

$$\lim_{n\to\infty}\sqrt{a+\sqrt{a+\cdots+\sqrt{a+\sqrt{b}}}},$$

the number of radicals being n.

b) Let $(a_n)_{n\geqslant 1}$ be a sequence of positive numbers and $(x_n)_{n\geqslant 1}$ the sequence defined by

$$x_n=\sqrt{a_n+\sqrt{a_{n-1}+\cdots+\sqrt{a_2+\sqrt{a_1}}}}.$$

Prove that:

(i) the sequence $(x_n)_{n\geqslant 1}$ is bounded if and only if the sequence $(a_n)_{n\geqslant 1}$ is bounded.

(ii) the sequence $(x_n)_{n\geqslant 1}$ is convergent if and only if the sequence $(a_n)_{n\geqslant 1}$ is convergent.

SOLUTION. a) We consider the sequence $(x_n)_{n\geqslant 1}$, where $x_1=\sqrt{b}$ and $x_{n+1}=\sqrt{a+x_n}$, for all $n\geqslant 1$. Since

$$x_{n+1}-x_{n+1}=\frac{x_{n+1}-x_n}{\sqrt{a+x_{n+1}}+\sqrt{a+x_n}},$$

it follows that the sequence is monotonic (increasing or decreasing).

In the case the sequence is decreasing, then it is bounded and convergent to some limit l. The limit satisfies $l^2-l-a=0$. Solving for l and using $l\geqslant 0$, we get $l=\frac{1+\sqrt{1+4a}}{2}$.

In the case $(x_n)_{n\geqslant 1}$ is increasing, observe that $\sqrt{a+x_n}=x_{n+1}\geqslant x_n$ which implies $x_n^2-x_n-a\leqslant 0$. Solving for x_n we get $x_n\leqslant\frac{1+\sqrt{1+4a}}{2}$. Therefore $(x_n)_{n\geqslant 1}$ is bounded in this case also.

b) Assuming $(x_n)_{n\geqslant 1}$ bounded, let $M>0$ be such that $x_n\leqslant M$ for all n. As $a_n\leqslant x_n\leqslant M$, it follows that $(a_n)_{n\geqslant 1}$ is bounded too.

Conversely, assume $(a_n)_{n\geqslant 1}$ bounded, that is $a_n \leqslant M$ for some positive M. Then $x_n \leqslant \sqrt{M+\sqrt{M+\cdots+\sqrt{M}}}$. Denote by y_n the last quantity. By the preceeding result, the sequence $(y_n)_{n\geqslant 1}$ is convergent, in particular bounded. It follows that $(x_n)_{n\geqslant 1}$ is also bounded.

Assume now that the sequence $(x_n)_{n\geqslant 1}$ is convergent. As $x_n = \sqrt{a_n + x_{n-1}}$, we obtain $a_n = x_n^2 - x_{n-1}$, implying that $(a_n)_{n\geqslant 1}$ is also convergent.

Converselly, assume that $(a_n)_{n\geqslant 1}$ is a converget sequence and let $a = \lim\limits_{n\to\infty} a_n$. Consider the following situations.

Suppose a is nonzero. We shall prove, that in this case $\lim\limits_{n\to\infty} x_n = \frac{1+\sqrt{1+4a}}{2}$. indeed, let ε be a positive number, such that $0 < \varepsilon < \sqrt{a}$, and let $A = a - \varepsilon^2$, $B = a + \varepsilon^2$. As a is the limit of a_n, one can find a positive integer $N = N_\varepsilon$, such that $A < a_n < B$, for any $n > N$. Consider the sequences defined by

$$A_n = \sqrt{A+\sqrt{A+\cdots+\sqrt{A}}}$$

and

$$B_n = \sqrt{B+\sqrt{B+\cdots+\sqrt{B+\sqrt{M+\cdots+\sqrt{M}}}}},$$

in both the total number of radicals being n and in the second the number of radicals that involve M being $n-N-1$.

By previous considerations

$$\lim_{n\to\infty} = \frac{1+\sqrt{1+4A}}{2} \text{ and } \lim_{n\to\infty} B_n = \frac{1+\sqrt{1+4B}}{2}.$$

Moreover we have $A_n < x_n < B_n$. The definitions of A and B imply the easy to check inequalities

$$\frac{1+\sqrt{1+4A}}{2} > \frac{1+\sqrt{1+4a}}{2} - \varepsilon,$$

and

$$\frac{1+\sqrt{1+4B}}{2} < \frac{1+\sqrt{1+4a}}{2} + \varepsilon.$$

We can therefore find $N'_\varepsilon = N'$, such that for all $n \geqslant N'$ the following inequalities are true

$$A_n \geqslant \frac{1+\sqrt{1+4a}}{2} - \varepsilon, \quad B_n \leqslant \frac{1+\sqrt{1+4a}}{2} + \varepsilon.$$

For $n \geqslant \max(N, N')$ we infer that

$$\frac{1+\sqrt{1+4a}}{2} - \varepsilon < A_n < x_n < B_n < \frac{1+\sqrt{1+4a}}{2} + \varepsilon.$$

These prove that $\lim\limits_{n\to\infty} x_n = \frac{1+\sqrt{1+4a}}{2}$.

In case $a = 0$, we shall prove that $\lim\limits_{n\to\infty} x_n = 1$. Indeed, it is obvious that $x_n > \sqrt[2^n]{a_1}$ and $\lim\limits_{n\to\infty} a_1^{\frac{1}{2^n}} = 1$. As in the preceeding considerations we get for some N_ε that $x_n < 1+\varepsilon$ for any $n > N$. This shows that $\lim\limits_{n\to\infty} x_n = 1$ in this case.

REMARK. The use of lim sup and lim inf can make the preeceding proof more elegant, but a little bit non-elementary.

PROBLEM 2. In a rectangular system of coordinates of a plane, we consider the points $A_n(n, n^3)$, where n runs over all positive integers and the point $B(0,1)$. Prove that:

a) for every integers $k \geqslant j > i \geqslant 1$ the points A_i, A_j, A_k are not on a line;

b) for every positive integers $1 \leqslant i_1 < i_2 < \cdots < i_{n-1} < i_n$, the following inequality holds:

$$\angle A_{i_1}OB + \angle A_{i_2}OB + \cdots + \angle A_{i_n}OB < \frac{\pi}{2}.$$

SOLUTION. a) For any three distinct points A_1, A_j, A_k, with $1 \leqslant i < jk$, the condition of colinearity is

$$\Delta = \begin{vmatrix} k & k^3 & 1 \\ j & j^3 & 1 \\ i & i^2 & 1 \end{vmatrix} = 0.$$

The value of Δ being $(k-j)(k-i)(j-i)(k+j+i)$, it is clear that $\Delta \neq 0$.

b) Let us denote $\angle A_iOB = x_i$, for any $i \geqslant 1$. We have $\tan x_i = \frac{i}{i^2} = \frac{1}{i^2}$. For $i = 1$ we get $x_1 = \frac{\pi}{4}$ and for $i > 1$, we deduce $x_i < \frac{\pi}{4}$. The inequality $x < \tan x$ $(x > 0)$, and the fact that $i_2 \geqslant 2, i_3 \geqslant 3, \ldots, i_n \geqslant n, \ldots$, implies

$$\begin{aligned}
\sum_{k=1}^{n} x_{i_k} &< \frac{\pi}{4} + \sum_{k=2}^{n} \tan x_{i_k} \leqslant \frac{\pi}{4} + \sum_{k=2}^{n} \frac{1}{i_k^2} \\
&\leqslant \frac{\pi}{4} + \sum_{k=2}^{n} \frac{1}{k^2} < \frac{\pi}{4} + \frac{1}{4} + \sum_{k=3}^{n} \frac{1}{k^2} \\
&< \frac{\pi}{4} + \frac{1}{4} + \sum_{k=3}^{n} \frac{1}{k(k+1)} \\
&= \frac{\pi}{4} + \frac{1}{4} + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&< \frac{\pi}{4} + \frac{3}{4} < \frac{\pi}{2}.
\end{aligned}$$

PROBLEM 3. a) Find a 3×3 matrix A with complex entries, $A \in M_3(\mathbf{C})$, such that $A^2 \neq 0$ and $A^3 = 0$.

b) Let n, p be numbers which are 2 or 3. We assume that there exists a function $f : M_n(\mathbf{C}) \to M_p(\mathbf{C})$ with properties:

- f is a bijective function;
- $f(XY) = f(X)\cdot f(Y)$, for every $X, Y \in M_n(\mathbf{C})$.

Prove that $n = p$.

SOLUTION. a) Take $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

b) Assume by contradiction that $n \neq p$. Since the inverse function f^{-1} is also multiplicative, we may suppose that $f : M_3(\mathbf{C}) \to M_2(\mathbf{C})$.

Taking $X = O_3$ and $Y \in M_3(\mathbf{C})$, such that $f(Y) = O_2$, we get $f(O_3) = O_2$. Consider now A as in a) and put $B = f(A)$. Then $B^3 = f(A^3) = f(O_3) = O_2$ and $B^2 = f(A^2) \neq f(O_3) = O_2$. We infer $B \in M_2(\mathbf{C})$, $B^3 = O_2$ and $B^2 \neq O_2$. But this is impossible, because $B^3 = O_2$ implies $\det B = 0$ and in turn $B^2 = (\operatorname{tr} B) \cdot B$ and $B^3 = (\operatorname{tr} B) \cdot B$, hence $\operatorname{tr} B = 0$ or $B = O_2$. In both cases $B^2 = O_2$, a contradiction.

PROBLEM 4. Let $f : \mathbf{R} \to \mathbf{R}$ be a function which satisfies the conditions:
(i) f has lateral limits in any point $a \in \mathbf{R}$ and

$$f(a-0) \leqslant f(a) \leqslant f(a+0);$$

(ii) for any real numbers a, b, $a < b$, one has

$$f(a-0) < f(b-0).$$

Prove that f is a monotonic increasing function.

SOLUTION. Take real numbers a, b such that $a < b$, and assume that $f(a) \geqslant f(b)$. By the hypothesis it follows that $f(a+0) \geqslant f(b-0)$.

If $f(a+0) > f(b-0)$, choose $y \in \mathbf{R}$ such that $f(a+0) > y > f(b-0)$, and $\varepsilon > 0$ such that $f(x) > y$ for any $x \in (a, a+\varepsilon)$ and $f(x) < y$ for any $x \in (b-\varepsilon, b)$. We may suppose $a + \varepsilon < b - \varepsilon$. Then, for $\alpha \in (a, a+\varepsilon)$ and $\beta \in (b-\varepsilon, b)$ we have $\alpha < \beta$ and $f(\alpha+0) \geqslant y \geqslant f(\beta - 0)$, which is a contradiction.

It remains the case $f(a+0) = f(b-0)$ which implies $f(a+0) = f(a) = f(b) = f(b-0)$. Take $c \in (a, b)$. Supposing $f(a+0) > f(c-0)$ we get in the same way as above a contradiction. Thus we have $f(a+0) \leqslant f(b-0)$. Analogously we obtain $f(c+0) \leqslant f(b-0)$. We collect $f(a+0) \leqslant f(c-0) \leqslant f(c) \leqslant f(c+0) \leqslant f(b-0)$, implying that f is constant on (a, b). It follows that for $\alpha, \beta \in (a, b)$, $\alpha < \beta$, we get $f(\alpha - 0) = f(\beta - 0)$, which contradicts the given condition.

REMARK FOR AN ALTERNATIVE SOLUTION. A more elegant proof can be obtained considering the same kind of arguments by using $l = \sup\{c \mid f(c) \geqslant f(b)\}$ under the supposition that $a < b$ and $f(a) \geqslant f(b)$.

12th GRADE

PROBLEM 1. Let A be a ring, $a \in A$, and n, k be integers such that $n \geqslant 2$, $k \geqslant 2$, $\underbrace{1+1+\ldots+1}_{n\text{ times}} = 0$ and $a^k = a + 1$. Prove that:

a) for every positive integer s, there exist non-negative integers $p_0, p_1, \ldots, p_{k-1}$ such that

$$a^s = p_0 \cdot 1 + p_1 \cdot a + \cdots + p_{k-1} \cdot a^{k-1};$$

b) there exists a pozitive integer m such that $a^m = 1$.

SOLUTION. a) We proceed by induction on s. For $s < k$ there is nothing to prove. For $s \geqslant k$, assume that

$$a^s = p_0 \cdot 1 + p_1 \cdot a + \cdots + p_{k-1} \cdot a,$$

and obtain

$$\begin{aligned} a^{s+1} &= a \cdot a^s = p_0 a + p_1 a^2 + \cdots + p_{k-1} a_k \\ &= p_k \cdot 1 + (p_0 + p_{k-1}) + p_1 a^2 + \cdots + p_{k-2} a^{k-1}. \end{aligned}$$

We observe that all computations are considered in the subring $\mathbf{Z}[a]$ of A, which is commutative.

b) Since $a^k = a + 1$ gives $a(a^{k-1} - 1) = 1$, we obtain that a is a unit of the ring $\mathbf{Z}[a]$. Moreover, since $n \cdot 1 = 0$ in $\mathbf{Z}[a]$, we conclude that the number of polynomial expressions

$$n_0 + n_1 a + \cdots + n_{k-1} a^{k-1} \in \mathbf{Z}[a]$$

is finite (it only involves the n_i with $0 \leqslant n_i < n$). Therefore, $\mathbf{Z}[a]$ is a finite ring. Any unit of $\mathbf{Z}[a]$ is an element of finite order in the group of units of $\mathbf{Z}[a]$, concluding thus the proof.

PROBLEM 2. a) For any positive integer n, let A_n be the ring

$$A_n = \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2 = \mathbf{Z}_2^n$$

Show that if $n \neq m$, then the rings A_n and A_m are not isomorphic but there exists a ringhomomorphism $f : A_n \to A_m$.

b) Prove that there exists rings $B_1, B_2, \ldots, B_n, \ldots$ such that, no homomorphism exists between B_n and B_m, whatever are the numbers $n \neq m$.

SOLUTION. a) The rings $\mathbf{Z}_2^m$ and $\mathbf{Z}_2^n$ are not isomorphic for $m \neq n$, as in that case they have different cardinalities.

For any n and all i, $1 \leqslant i \leqslant n$, consider the projections $p_i : \mathbf{Z}_2^n \to \mathbf{Z}_2$, given by $p_i(a_1, a_2, \ldots, a_n) = a_i$, which are also ring homomorphisms. Define, for any m the ring homomorphism $\lambda : \mathbf{Z}_2 \to \mathbf{Z}_2^m$, by $\lambda(0) = (0, 0, \ldots, 0)$ and $\lambda(1) = (1, 1, \ldots, 1)$. The composition of functions $\mathbf{Z}_2^n \to \mathbf{Z}_2 \to \mathbf{Z}_2^m$, is a ring homomorphism.

b) We can get several examples as required.

Let $p_1 < p_2 < \cdots < p_n < \cdots$ be the sequence of prime numbers. The fields $B_i = \mathbf{Z}_{p_i}$ are not connected by any morphism, as being of different characteristics.

Another example is: take $B_i = \mathbf{Q}(\sqrt{p_i})$. There are no homomorphisms between them, like the following lemma concludes.

LEMMA. Let p, q be distinct prime numbers. There is no ring homomorphism $f : \mathbf{Q}(\sqrt{p}) \to \mathbf{Q}(\sqrt{q})$.

PROOF. If f would be such a homomorphism, then $f(a) = a$ for any $a \in \mathbf{Q}$. Supposing $f(\sqrt{p}) = a + b\sqrt{q}$, with $a, b \in \mathbf{Q}$, from $(\sqrt{p})^2 = p$ we obtain by the morphism condition $f(\sqrt{p})^2 = f(p)$, and thus $(a + b\sqrt{q})^2 = p$. Therefore $2ab\sqrt{q} = p - a^2 - b^2 q$. One cannot have $a = 0$ or $b = 0$, because this should imply that $\frac{p}{q}$ or p are perfect squares in $\mathbf{Q}$. We conclude $ab \neq 0$ and therefore $\sqrt{q} = \frac{p - a^2 - b^2 q}{2ab} \in \mathbf{Q}$ which is a contradiction.

PROBLEM 3. For any real number a, $0 < a \leqslant 1$, we denote

$$I_n(a) = \int_0^a \ln(1 + x + \cdots + x^{n-1})\,\mathrm{d}x, \quad n \geqslant 2.$$

Compute the limit: $\lim_{n\to\infty} I_n(a)$.

SOLUTION. Consider the following two cases: $a<1$ and $a=1$.

If $a<1$, standard computations lead to

$$I_n(a)=\int_0^1 \ln\frac{1-x^n}{1-x}\mathrm{d}x.$$

Since

$$\frac{1-a^n}{1-x}\leqslant\frac{1-x^n}{1-x}<\frac{1}{1-x},$$

for $x\in[0,a]$, we obtain

$$\int_0^a \ln\frac{1-a^n}{1-x}\mathrm{d}x\leqslant I_n(a)\leqslant\int_0^a \ln\frac{1}{1-x}\mathrm{d}x.$$

Computation of the integrals leads to the inequalities

(1) $$(1-a^n)[a+(1-a)\ln(1-a)]\leqslant I_n(a)\leqslant a+(1-a)\ln(1-a).$$

As $\lim_{n\to\infty} a^n=0$, we conclude

$$\lim_{n\to\infty} I_n(a)=a+(1-a)\ln(1-a).$$

Consider the case $a=1$. We shall prove that $\lim_{n\to\infty} I_n(1)=1$. Observe first, that for $a<1$ the sequence $I_n(a)$ is increasing. The above result implies then

$$I_n(a)\leqslant a+(1-a)\ln(1-a)$$

for all $n\geqslant 2$. For any $n\geqslant 2$, the function $I_n:[0,1]\to[0,\infty)$, given by

$$I_n(a)=\int_0^a \ln(1+x+\cdots+x^{n-1})\mathrm{d}x,$$

is continuous at 1. We conclude, therefore, that

$$0\leqslant I_n(1)=\lim_{a\to1} I_n(a)\leqslant\lim_{a\to1}\left(a+(1-a)\ln(1-a)\right),$$

for any $n\geqslant 2$. As the sequence $I_n(1)$ is increasing and bounded, it is convergent. Denote $l=\lim_{n\to\infty} I_n(1)$. From

$$I_n(a)=I_n(1)-\int_a^1 \ln(1+x+\cdots+x^{n-1})\mathrm{d}x\leqslant I_n(1),$$

we obtain, $I_n(a)\leqslant I_nn(1)\leqslant 1$. Taking limits as $n\to\infty$, one derives $a+(1-a)\ln(1-a)\leqslant l\leqslant 1$, for any $a\in(0,1)$. For $a\to1$, we finally conclude $l=1$.

ALTERNATIVE SOLUTION. Consider the double inequality easily obtained in the previous calculations, for any $a<1$:

(2) $$I_n(a)\leqslant I_n(1)\leqslant I_n(a)+(1-a)\ln n.$$

Put $a=a_n=1-\frac{\ln n}{n}$. It easy to see that $a_n\to1$ and $1-a_n^n\to1$ when $n\to\infty$. By inequality (1) from part a), taking limits as $n\to\infty$ in (2) we derive $\lim_{n\to\infty} I_n(1)=1$.

PROBLEM 4. Let $f:\mathbf{R}\to[0,\infty)$ be a continuous function which is periodic of period 1. Prove that:

a) $\displaystyle\int_a^{a+1} f(x)\,\mathrm{d}x=\int_0^1 f(x)\,\mathrm{d}x,\ \forall a\in\mathbf{R}$;

b) $\displaystyle\lim_{n\to\infty}\int_0^1 f(x)f(nx)\,\mathrm{d}x=\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$.

SOLUTION. a) Consider an integer n such that $a\leqslant n$. Then

$$\begin{aligned}\int_a^{a+1} f(x)\mathrm{d}x&=\int_a^n f(x)\mathrm{d}x+\int_n^{a+1} f(x)\mathrm{d}x\\&=\int_{a-n+1}^1 f(t+n-1)\mathrm{d}t+\int_0^{a+1-n} f(t+n)\mathrm{d}t\\&=\int_{a-n+1}^1 f(t)\mathrm{d}t+\int_0^{a-n+1} f(t)\mathrm{d}t=\int_0^1 f(t)\mathrm{d}t.\end{aligned}$$

b) We have

$$\int_0^1 f(x)f(nx)\mathrm{d}x=\frac{1}{n}\int_0^n f\left(\frac{t}{n}\right)f(t)\mathrm{d}t=\frac{1}{n}\sum_{k=0}^{n-1}\int_k^{k+1} f\left(\frac{t}{n}\right)f(t)\mathrm{d}t.$$

The mean value property shows that there exist points c_k, $\frac{k}{n}\leqslant c_k\leqslant\frac{k+1}{n}$, such that:

$$\int_0^1 f(x)f(nx)\mathrm{d}x=\frac{1}{n}\sum_{k=0}^{n-1}\int_k^{k+1} f(c_k)f(t)\mathrm{d}t=\frac{1}{n}\sum_{k=o}^{n-1} f(c_k)\int_0^1 f(t)\mathrm{d}t.$$

Since $\sum_{k=0}^{n-1} f(c_k)$ is a Riemann sum for f on the interval $[0,1]$, we conclude

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} f(c_k)=\int_0^1 f(t)\mathrm{d}t,$$

from where the result easily follows.

II.3. FINAL ROUND

7th GRADE

PROBLEM 1. Eight card players are seated around a table. One remarks that at some moment, any player and his two neighbours have alltogether an odd number of winning cards.

Show that any player has at that moment at least one winning card.

SOLUTION. We shall prove more: each player has an odd number of winning cards. Denote the players disposed around the table and identify them with numbers: $1, 2, 3, 4, 5, 6, 7, 8, 1, 2, \ldots$. We shall attach to each player the number 0 if the number of winning cards he has is even and the number 1 otherwise.

One obtains a sequence of zeroes and ones such that to any consecutive three numbers the number 1 is attached once or to all three.

If the number 0 is attached to a player, say player number 1 than only the following possibilities can occur:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{matrix} \quad \text{or} \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{matrix}.$$

In both cases players 8, 1 and 2 have alltogeather an even number of winning cards, and this is a contradiction.

PROBLEM 2. Prove that any real number x, $0 < x < 1$ can be written as a difference of two positive and less than 1 irrational numbers.

SOLUTION. Let x be an arbitrary real number such that $0 < x < 1$.
Consider the cases:

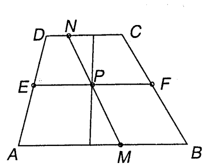
a) $x \in \mathbf{Q}$. Let $x_0 \in \mathbf{R}\setminus\mathbf{Q}$, $x_0 > 0$. One can find $n \in \mathbf{N}^*$ such that $x + \frac{x_0}{n} < 1$. Denoting $y_2 = x + \frac{x_0}{n}$, we have $y_2 \in \mathbf{R}\setminus\mathbf{Q}$, $0 < y_2 < 1$. If $y_1 = \frac{x_0}{n}$ then evidently $y_1 \in \mathbf{R}\setminus\mathbf{Q}$ and $0 < y_1 < 1$. It is clear that $x = y_2 - y_1$.

b) $x \in \mathbf{R}\setminus\mathbf{Q}$. Consider $n \in \mathbf{N}^*$ such that $x + \frac{x}{n} < 1$. Consider $y_2 = x + \frac{x}{n}$ and $y_1 = \frac{x}{n}$. Then $x = y_2 - y_1$ and $y_1, y_2 \in \mathbf{R}\setminus\mathbf{Q}$.

PROBLEM 3. Let $ABCD$ be a trapezoid and AB respectively CD be its parallel edges. Find, with proof, the set of interior points P of the trapezoid which have the following property:

"P belongs to at least two lines each intersecting the segments AB and CD and each dividing the trapezoid in two other trapezoids with equal areas".

SOLUTION. We shall prove that the only point P that satisfies the condition is the midpoint of the median. Denote by M, N respectively the midpoints of AD and BC.



If a line d having the given property, intersects CD and AB in E and F respectively, than using the area formula for a trapezoid, $DE + AF = CE + BF$ which is equivalent to $DE + AF = CD - DE + AB - AF$, or $DE + AF = \frac{AB+CD}{2} = MN$.

If two such lines d and d' intersect at P and have the given property, then, denoting with E' and F' the intersection of d' with CD and AB respectively, the former relation imply that the segments determinated by E, E' on DC and F, F' on AB are congruent, that is the triangles PEE' and PFF' are congruent. This easily implies that P belongs to MN.

If $P \in MN$, from $DE + AF = MN$ we conclude $MP = MN/2$, that is P is the midpoint of MN.

Reciprocally, it is trivial to see that the midpoint of MN satisfies the given condition.

PROBLEM 4. a) An equilateral triangle of sides a is given and a triangle MNP is constructed under the following conditions: $P \in (AB)$, $M \in (BC)$, $N \in (AC)$, such that $MP \perp AB$, $NM \perp BC$ and $PN \perp AC$. Find the lenght of the segment MP.

b) Show that for any acute triangle ABC one can find points $P \in (AB)$, $M \in (BC)$, $N \in (AC)$, such that $MP \perp AB$, $NM \perp BC$ and $PN \perp AC$.

SOLUTION. a) Having equal angles, the triangles NPM and ABC are similar, implying that ΔNPM is also equilateral. This means $\Delta APN \equiv \Delta BMP \equiv \Delta CNM$. As $AP = x$ we have $AN = BP = a - x$ and $x = \frac{a-x}{2}$ that is $x = \frac{a}{3}$. In the right triangle APN the Pitagorean formula gives $PN = \frac{a\sqrt{3}}{3}$.

b) Let P_0 be an arbitrary point on AB, N_0 its projection on AC and M_0 be the intersection between the perpendicular dropped from N_0 to BC and the perpendicular raised from P_0 to AB. Suppose AM_0 intersects BC at M. The homothety of center A and ratio $\frac{AM}{AM_0}$ sends the triangle $M_0N_0P_0$ to the triangle MNP with the desired properties.

8th GRADE

PROBLEM 1. For any number $n \in \mathbf{N}$, $n \geqslant 2$, denote by $P(n)$ the number of pairs (a, b) whose elements are of positive integers such that

$$\frac{n}{a} \in (0,1), \quad \frac{a}{b} \in (1,2) \quad \text{and} \quad \frac{b}{n} \in (2,3).$$

a) Calculate $P(3)$.

b) Find n such that $P(n) = 2002$.

SOLUTION. From $\frac{n}{a} \in (0,1)$, $\frac{a}{b} \in (1,2)$, $\frac{b}{n} \in (2,3)$ we infer $2n < b < a < 2b < 6n$.

Easily $2n < b < 3n$ implies $b \in \{2n+1, \ldots, 3n-1\}$, that is b can take only $n-1$ different values. For each such b we have $a \in \{b+1, \ldots, 2b-1\}$, that is a can take only $b-1$ distinct values.

For $b = 2n+1$ we obtain $2n$ distinct values for a, collecting thus $2n$ pairs (a, b). In the same way for $b = 2n+2$ we obtain $2n+1$ pairs, a.s.o. and for $b = 3n-1$ we have $3n-2$ pairs. Summing up

$$P(n) = 2n + (2n+1) + \cdots + (3n-2) = \frac{(n-1)(5n-2)}{2}.$$

In particular $P(3) = 13$. For $P(n) = 2002$ we easily find $n = 29$.

PROBLEM 2. Given real numbers a, c, d, show that there exists at most one function $f : \mathbf{R} \to \mathbf{R}$ which satisfies:

$$f(ax + c) + d \leqslant x \leqslant f(x + d) + c, \text{ for any } x \in \mathbf{R}.$$

SOLUTION. If $a = 0$ we would have $x \geqslant d + f(c)$, for any real x, a contradiction.

Suppose $a \neq 0$ and let $ax + c = y$. The first inequality then yields

$$f(y) \leqslant \frac{y}{a} - \frac{ad + c}{a},$$

for any $y \in \mathbf{R}$.

If we substitute $y = x + d$ in the second inequality, we get $f(y) \geqslant y - d - c$, for any real y. The two obtained relations imply

$$(a - 1)\left(\frac{y - c}{a}\right) \geqslant 0,$$

for all $y \in \mathbf{R}$. As $y - \frac{c}{a}$ takes positive and negative values, we conclude $a = 1$.

Hence $f(y) = y - d - c$ is the answer of the problem.

PROBLEM 3. Let $[ABCA'B'C']$ be a frustum of a regular pyramid. Let G and G' be the centroids of bases ABC and $A'B'C'$ respectively. It is known that $AB = 36$, $A'B' = 12$ and $GG' = 35$.

a) Prove that the planes $(ABC'), (BCA'), (CAB')$ have a common point P, and the planes $(A'B'C), (B'C'A), (C'A'B)$ have a common point P', both situated on GG'.

b) Find the length of the segment $[PP']$.

SOLUTION. a) Let N and N' be the midpoints of the segments BC and $B'C'$ respectively. Suppose $A'N$ and GG' intersect at P. Triangles $A'PG'$ and NPG are similar, hence

$$\frac{G'P}{GP} = \frac{G'A'}{GN} = 2\frac{A'N'}{AN} = \frac{2}{3}.$$

Since $A'N \subset (A'BC)$ we deduce that the plane $(A'BC)$ intersects the segment GG' at the point P such that

$$\frac{G'P}{GP} = \frac{2}{3}.$$

Similar arguments show that planes (BCA') and (CAB') also pass through P. In the same way we can show that planes $(AB'C')$, $(BC'A')$ and $(CB'A')$ pass through the point P' on GG' such that $G'P' = 5$.

b) Let M be the projection of A' onto (ABC). Obviously $M \in AA_1$ and $A'M = GG'$. By the similarity of the triangles PC_1A_1 with $PC'A'$ and PGA_1 with $A'MA_1$ we get

$$\frac{PG}{GG'} = \frac{PA_1}{A'A_1} = \frac{A_1C_1}{A'C'} = \frac{\frac{AC}{2}}{A'C'} = \frac{18}{18 + 12}.$$

From $GG' = 35$ we obtain $PG = 21$. Analogously, it is easy to find $P'G' = 5$. Finally $PP' = 35 - 21 - 5 = 9$.

PROBLEM 4. The right prism $[A_1A_2A_3 \cdots A_nA'_1A'_2 \cdots A'_n]$, $n \in \mathbf{N}$, $n \geqslant 3$, has a convex polygon as its base. It is known that $A_1A'_2 \perp A_2A'_3, A_2A'_3 \perp A_3A'_4, \ldots, A_{n-1}A'_n \perp A_nA'_1, A_nA'_1 \perp A_1A'_2$. Show that:

a) $n = 3$;

b) the prism is regular.

SOLUTION. a) If we consider the point B_3, symmetrical to A_3 with respect to A_2, we notice that the equadrilateral $A_2A'_3A'_2B_3$ is a parallelogram, hence A'_2B_3 is parallel to $A_2A'_3$. It follows that $\angle A_1A'_2B_3 = 90°$. Since A_2 is the projection of A'_2 on the base plane, it results that $\angle A_2A_2B_3 > 90°$, thus $\angle A_1A_2A_3 < 90°$. Similar arguments show that all the angles of the base polygon are acute. This yields $180°(n-2) < 90°n$, or $n < 4$, hence $n = 3$.

b) Let B_4 a point such that the segment A_2B_4 is parallel and equal to A_1A_3. It is not difficult to find that the sides A'_2A_1, A'_2B_3 and A'_2B_4, of the tetraedron $A'_2A_1B_3B_4$ are mutuallly perpendicular, hence the projection of A'_2 on the plane $A_1B_3B_4$ – namely the point A_2 – is the orthocenter of the triangle $A_1B_3B_4$. Since $A_1A_2B_4A_3$ is a parallelogram, B_3A_2 is a median in the triangle $A_1B_3B_4$, hence $A_2A_1 = A_2B_4 = A_2B_3$. It follows that $A_1A_2A_3$ is an equilateral triangle. By symmetry we arrive at the given conclusion.

9th GRADE

PROBLEM 1. Let a, b, c be positive numbers such that $ab + bc + ca = 1$. Show that:

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geqslant \sqrt{3} + \frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}.$$

SOLUTION. Solving $ab + bc + ca = 1$ for c, we deduce $c(a+b) = 1 - ab$ and

$$c = \frac{1}{a+b} - \frac{ab}{a+b},$$

and the similar ones.

Summing up, we find

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} - \frac{ab}{a+b} - \frac{bc}{a+b} - \frac{ac}{a+c} = a + b + c.$$

As $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca) \geqslant 3(ab+bc+ca) = 3$ by a well-known inequality, we conclude the proof.

PROBLEM 2. Let ABC be a right triangle $\angle A = 90°$ and let $M \in AB$ such that $\frac{AM}{MB} = 3\sqrt{3} - 4$. It is known that the symmetric point of M with respect to the line GI lies on AC. Find the measure of angle B (*G is the centroid and I is the center of the incircle*).

SOLUTION. We shall use vectors. Let N be the symmetric point of M with respect to the midpoint of GI. Consider α and β such that $\overrightarrow{AM} = \alpha\overrightarrow{AB}$, $\overrightarrow{AN} = \beta\overrightarrow{AC}$, $\overrightarrow{GI} = \overrightarrow{GM} + \overrightarrow{GN} = (2 - 2\alpha - \beta)\overrightarrow{GA} + (\beta - \alpha)\overrightarrow{GC}$.

On the other side

$$\overrightarrow{GI} = \frac{(a-b)\overrightarrow{GA} + (c-b)\overrightarrow{GC}}{a+b+c}.$$

By identifying coefficients, we get $\alpha = \dfrac{1}{3} + \dfrac{b}{a+b+c} = \dfrac{1}{3} + \dfrac{3-\sqrt{3}}{6}$.

Denote $x = b/a$, $y = c/a$. One obtains

$$\begin{cases} x^2 + y^2 = 1 \\ \dfrac{x}{1+x+y} = \dfrac{3-\sqrt{3}}{6}. \end{cases}$$

Solving for x, we get $x = 1/2$, implying therefore $m(\angle B) = 30°$.

PROBLEM 3. Let k and n be positive integers with $n > 2$. Show that the equation:

$$x^n - y^n = 2^k$$

has no positive integer solutions.

SOLUTION. We shall proceed by contradiction. Let $n_0 > 2$ be minimal for which there exists $m > 0$ such that and $x^{n_0} - y^{n_0} = 2^m$. If n_0 is even, say $n_0 = 2k$, $k \in \mathbf{N}$, then by decomposition, $x^{2k} - y^{2k} = (x^k - y^k)(x^k + y^k)$, we conclude that $x^k - y^k$ is a power of 2. This contradicts the minimality condition.

It follows that n_0 is odd. Define the set

$$A = \{p \in \mathbf{N}^* \mid \text{there are } x, y \in \mathbf{N}^*, \text{ with } x^{n_0} - y^{n_0} = 2^p\}.$$

Let p_0 be the minimal element in A. If $x^{n_0} - y^{n_0} = 2^{p_0}$, it follows that x, y have the same parity. As $(x-y)(x^{n_0-1} + \cdots + y^{n_0-1}) = 2^{p_0}$, we deduce that x and y are even.

Consider $x = 2x_1, y = 2y_1$. It follows $x_1^{n_0} - y_1^{n_0} = 2^{p_0 - n_0}$, contradicting the minimality of p_0 or $x_1^{n_0} - y_1^{n_0} = 1$. It is trivial to see that the last equation has no positive integer solutions for $n_0 > 2$.

PROBLEM 4. Find all functions $f : \mathbf{N} \to \mathbf{N}$ which satisfy the equality

$$f(3x + 2y) = f(x)f(y),$$

for all $x, y \in \mathbf{N}$.

SOLUTION. For $x = y = 0$ we have $f(0) = f(0)^2$, that is $f(0) = 0$ or $f(0) = 1$.

The case $f(0) = 0$ gives $f(2y) = f(3x)$ for any $x, y \in \mathbf{N}$. Letting $f(1) = a$, we obtain $f(5) = f(3 \cdot 1 + 2 \cdot 1) = a^2$. In the same way, $f(25) = a^4$. On the other side $f(25) = f(2 \cdot 2 + 3 \cdot 7) = 0$, implying $a = 0$.

Because any integer $k > 4$ can be written in the form $k = 3x + 2y$, for some integers x, y, we find out $f(k) = 0$ for all k.

For the choice $f(0) = 1$ we have $f(2y) = f(y)$, $f(3x) = f(x)$ and letting $f(1) = a$ we derive $f(2) = a$, $f(5) = a^2$, $f(25) = a^3 = a^4$. These relations lead to $a = 0$ or $a = 1$.

In conclusion, we obtain the following functions:

$$f(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x > 0 \end{cases}$$

or $f(x) = 1$ for all $x \in \mathbf{N}$.

10th GRADE

PROBLEM 1. Let X, Y, Z, T be four points in the plane. The segments $[XY]$ şi $[ZT]$ are said to be *connected*, if there is some point O in the plane such that the triangles OXY and OZT are rightangled in O and isosceles.

Let $ABCDEF$ be a convex hexagon such that the pairs of segments $[AB]$, $[CE]$, and $[BD]$, $[EF]$ are *connected*. Show that the points A, C, D and F are the vertices of a parallelogram and that the segments $[BC]$ and $[AE]$ are *connected*.

SOLUTION. Suppose that the triangles OXY are OZT are counterclockwise oriented, and let x, y, z, t be the afixes of the points X, Y, Z, T and let m be the afix of O. As these triangles are right and isosceles we have $x - m = \mathrm{i}(y - m), z - m = \mathrm{i}(t - m)$. It follows $m(1 - \mathrm{i}) = x - \mathrm{i}y = z - \mathrm{i}t$. We deduce $x - z = \mathrm{i}(y - t)$.

Reciprocally, if $x - \mathrm{i}y = z - \mathrm{i}t$, the afix of O is $m = \frac{x - \mathrm{i}y}{1 - \mathrm{i}}$, and the triangles OXY and OZT are right and isosceles.

Let a, b, c, d, e, f be the afixes of the given hexagon in that order. We can write $a - \mathrm{i}b = c - \mathrm{i}e, b - \mathrm{i}d = e - \mathrm{i}f$. It follows $a + d = c + f$, that is $ACDF$ is a parallelogram.

Multiplying the first equality by i, we obtain $b - \mathrm{i}c = e - \mathrm{i}a$, that is BC and AE are connected.

ALTERNATIVE SOLUTION. We shall consider geometrical transformations. Let O_1 be the common vertex of the right isosceles triangles O_1AB and O_1CE and let O_2 be the common point of the triangles O_2BD and O_2EF. If R_i denotes the rotation with center O_i and angle $\frac{\pi}{2}$, then

$$A = R_1(B), \quad B = R_2(D), \quad \text{that is } A = (R_1 \circ R_2)(D).$$

Analogously

$$E = R_1^{-1}(C), \quad F = R_2^{-1}(E), \quad \text{that is } F = (R_2^{-1} \circ R_1^{-1})(C) = (R_1 \circ R_2)^{-1}(C).$$

Remark that $R_1 \circ R_2$ is a rotation of angle π. This implies $(R_1 \circ R_2)^{-1} = R_1 \circ R_2$, and by consequence, A and F are obtained from D and C, respectively, by the same rotation of angle π. We conclude that $ACDF$ is a parallelogram.

PROBLEM 2. Find all real polynomials f and g, such that:

$$(x^2 + x + 1) \cdot f(x^2 - x + 1) = (x^2 - x + 1) \cdot g(x^2 + x + 1),$$

for all $x \in \mathbf{R}$.

SOLUTION. Put $f(X) = Xa(X)$, $g(X) = Xb(X)$. For ω a non-real cubic root of 1, we get (as $\omega^2 + \omega + 1 = 0$) $g(0) = 0$ and for α a nonreal root of -1 we get

$f(0) = 0$. This proves that $a(X)$ and $b(X)$ are real polynomials. The condition in the hypothesis simplifies to $a(x^2 - x + 1) = b(x^2 + x + 1)$.

Changing x with $-x$ we get $a(x^2 + x + 1) = b(x^2 - x + 1)$. As $a(x^2 - x + 1) = b(x^2 + x + 1) = b((x + 1)^2 - (x + 1) + 1) = a((x + 1)^2 + (x + 1) + 1)$ we find out

$$a(x^2 - x + 1) = a((x + 1)^2 + (x + 1) + 1). \tag{1}$$

We shall prove by induction that $a(n^2 + 3n + 3) = a(1)$, for any integer n.

Indeed, for $x = 1$ the previous equality gives $a(1) = a(7)$. Supposing $a(n^2 + 3n + 3) = a(1)$, we write

$$\begin{aligned} a\left((n + 1)^2 + 3(n + 1) + 3\right) &= a\left((n + 2)^2 + (n + 2) + 1\right) \\ &= a\left((n + 1)^2 - (n + 1) + 1\right) = a(n^2 + n + 1), \end{aligned}$$

by (1), finishing thus the induction step. But all numbers n^2+n+1 are all mutually different, that is $a(X)$ is the constant polynomial. In conclusion $f(X) = kX$, where k is a real constant. This, in turn, easily implies $g(X) = kX$.

PROBLEM 3. Find all real numbers a, b, c, d, e in the interval $[-2, 2]$, that satisfy:

$$\begin{aligned} a + b + c + d + e &= 0 \\ a^3 + b^3 + c^3 + d^3 + e^3 &= 0 \\ a^5 + b^5 + c^5 + d^5 + e^5 &= 10 \end{aligned}$$

SOLUTION. It is natural to use the following substitutions $a = 2\cos x$, $b = 2\cos y$, $c = 2\cos z$, $d = 2\cos t$, $e = 2\cos u$.

By standard formulas $2\cos 5x = (2\cos x)^5 - 5(2\cos x)^3 + 5(2\cos x) = a^5 - 5a^3 + 5a$.

It follows

$$\sum 2\cos 5x = \sum a^5 - 5\sum a^3 + 5\sum a = 10.$$

that is $\sum \cos 5x = 5$, implying $\cos 5x = \cos 5y = \cos 5z = \cos 5t = \cos 5u = 1$. The relation $\cos 5\alpha = 1$ is equivalent to $5\alpha = 2k\pi$, for integer k. Since $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{2}$, $\cos\frac{4\pi}{5} = -\frac{\sqrt{5}+1}{2}$, $\cos 0 = 1$, we conclude $a, b, c, d, e \in \{2, \frac{\sqrt{5}-1}{2}, -\frac{\sqrt{5}+1}{2}\}$.

As $\sum a = 0$ it is clear that one of the numbers must be 2, other two equal $\frac{\sqrt{5}-1}{2}$ and the other equal $-\frac{\sqrt{5}+1}{2}$.

One checks that in this case the equality $\sum a^3 = 0$ is also fullfiled.

PROBLEM 4. Let $I \subseteq \mathbf{R}$ be an interval and $f : I \to \mathbf{R}$ a function such that:

$$|f(x) - f(y)| \leqslant |x - y|, \text{ for all } x, y \in I.$$

Show that f is monotonic on I if and only if, for any $x, y \in I$, either $f(x) \leqslant f\left(\frac{x+y}{2}\right) \leqslant f(y)$ or $f(y) \leqslant f\left(\frac{x+y}{2}\right) \leqslant f(x)$.

SOLUTION. Suppose, by contradiction, that f fails to be monotone. In that case one can find $x < y < z$ $(x, y, z \in I)$ such that $f(x) < f(y) > f(z)$ or $f(x) > f(y) < f(z)$.

By symmetry, it is sufficient to consider the case $f(x) < f(y) > f(z)$. Let $\lambda \in \mathbf{R}$ be such that $f(x) < \lambda$, $f(z) < \lambda$ and $\lambda < f(y)$. Consider the midpoint of $I_0 = [x, z]$. The point y is situated in one of the half intervals determined by this midpoint. Denote it by I_1. As $f(\frac{x+z}{2}) < \lambda$, the values of f at the endpoints of I_1 are less than λ. Consider the midpoint of I_1 and I_2 be the subinterval that contains y a.s.o. Inductively, we can find intervals $I_n = [a_n, b_n]$, $n \in \mathbf{N}^*$ having the property: $f(a_n) < \lambda$, $f(b_n) < \lambda$, $y \in I_n$, $b_n - a_n = \frac{z-x}{2^n}$. The inequality in the hypothesis implies

$$|f(y) - f(a_n)| \leqslant |a_n - y| < \frac{(z - x)}{2^n},$$

that is $f(y) < \lambda + \frac{z-x}{2^n}$, $\forall n \in \mathbf{N}^*$.

As $\lambda < f(y)$ we can find a positive integer n_0 such that $\lambda + \frac{z-x}{2^{n_0}} < f(y)$, which contradicts the previous inequality.

11th GRADE

PROBLEM 1. In the Carthesian plane xOy consider the hyperbola

$$\Gamma = \left\{M(x, y) \in \mathbf{R}^2 \,\middle|\, \frac{x^2}{4} - y^2 = 1\right\}$$

and a conic Γ', disjoint from Γ. Let $n(\Gamma, \Gamma')$ be the maximal number of pairs of points $(A, A') \in \Gamma \times \Gamma'$ such that $AA' \leqslant BB'$, for any $(B, B') \in \Gamma \times \Gamma'$.

For each $p \in \{0, 1, 2, 4\}$, find, the equation of Γ' for which $n(\Gamma, \Gamma') = p$. Justify the answer.

(The following curves are considered here as conics: *the circle, the ellipse, the hyperbola and the parabola.*)

SOLUTION.

Case $p = 0$. Let Γ' be the hyperbola given by the equation $\frac{x^2}{4} - y^2 + 1 = 0$. Choosing the points $B\left(n, \sqrt{\frac{n^2}{4} - 1}\right) \in \Gamma$, $B'\left(n, \sqrt{\frac{n^2}{4} - 2}\right) \in \Gamma'$, we have by an easy computation $BB' < \frac{2}{n}$. Since $BB' \to 0$ when $n \to \infty$, it follows $n(\Gamma, \Gamma') = 0$.

Case $p = 1$. Let Γ' be the circles given by the equation $x^2 - x + y^2 = 0$, and let $A(2.0)$, $A'(1.0)$. Let $(M.M') \in (\Gamma \times \Gamma')$ and N, N' the intersection points between MM' and the tangents to Γ, Γ' at A, A'. It is not difficult to see that $AA' \leqslant NN' \leqslant MM'$, with equality if and only if $A = M$ and $A' = M'$. Thus $n(\Gamma, \Gamma') = 1$.

Case $p = 2$. Let Γ' be the circle of equation $x^2 + y^2 = 1$. Points $A_1(2, 0)$, $A_1'(1, 0)$, $A_2(-2, 0)$ and $A_2'(-1, 0)$ can be used to prove in a similar way that $n(\Gamma, \Gamma') = 2$.

Case $p = 4$. Consider Γ', the hyperbola of equation $\frac{y^2}{4} - x^2 = 1$ and the points $A_1\left(\frac{4\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$, $A_1'\left(\frac{\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right)$, $A_2\left(\frac{-4\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$, $A_2'\left(\frac{-\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right)$, $A_3\left(\frac{-4\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)$, $A_3'\left(\frac{-\sqrt{3}}{3}, \frac{-4\sqrt{3}}{3}\right)$, $A_4\left(\frac{4\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)$, $A_4'\left(\frac{\sqrt{3}}{3}, \frac{-4\sqrt{3}}{3}\right)$. In this case $n(\Gamma, \Gamma') = 4$.

PROBLEM 2. Let $f : \mathbf{R} \to \mathbf{R}$ be a function that has limits at any point and has no local extrema. Show that:

a) f is continuous;
b) f is strictly monotone.

SOLUTION. a) Let $x_0 \in \mathbf{R}$ be arbitrary in $\mathbf{R}$. If $f(x_0) < \lim\limits_{x\to x_0} f(x)$ then x_0 is a local minimum point and if $f(x_0) > \lim\limits_{x\to x_0} f(x)$, then x_0 is a local maximum point. Both cases contradict the given condition. It follows that f is continuous at x_0.

b) We shall prove that f is one-to-one. If not, let $a, b,\ a < b$ such that $f(a) = f(b)$. As f is not constant on $[a, b]$ (otherwise any point $c \in (a, b)$ is a local extremum point), it follows that one can find an extremum point $d \in (a, b)$, and we get a contradiction. We conclude that f is one-to-one. Being continuous, the intermediate value property implies that f is strictly monotone.

PROBLEM 3. Let $A \in M_4(\mathbf{C})$ a non-zero matrix.

a) If $\operatorname{rank}(A) = r < 4$, prove the existence of two invertible matrices $U, V \in M_4(\mathbf{C})$, such that:

$$UAV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the r-unit matrix.

b) Show that if A and A^2 have the same rank k, then the matrix A^n has rank k, for any $n \geqslant 3$.

SOLUTION. We shall consider the general case when $A \in M_m(\mathbf{C})$, $m \in \mathbf{N}^*$.

a) Let $i \neq j \in \{1, \ldots, m\}$ and $a \in \mathbf{C}^*$.

Consider the matrices P_{ij} obtained from the identity I_m by permuting rows i and j; the matrices $T_i(a)$, obtained from I_m by multiplying row i by a and $S_{ij}(a)$, derived from I_m by adding row i multiplied by a, to row j. All these matrices are invertible.

To multiply A to the left by these matrices, corresponds to the following operations, respectively: permute rows i and j; multiplying row i by a, and adding to row j the row i multiplyied by a. Multiplying A to the right by the same matrices, coresponds to similar operations on the columns of A. It follows that by a sequence of such multiplications to the left and/or right, one should obtain a matrix of the required form. The matrix U will be the product of all matrices used in the multiplication to the left and V represents the product of the sequence of matrices used in the multiplication to the right of A.

b) If $\operatorname{rank}(A) = m$ then $\det(A) \neq 0$. It follows that $\det(A^n) \neq 0$, that is $\operatorname{rank}(A^n) = m$.

Let $k = \operatorname{rang}(A) < m$. By using a) we have $UAV = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, where U and V invertible. As $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} = C \cdot D$, where $C = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$, $D = (I_k \quad 0)$, and $A = U^{-1}C \cdot DV^{-1}$ we obtain $A = EF$, where $E = U^{-1}C$ and $F = D \cdot V^{-1}$, both having rank k.

From $A^2 = EFEF$ we deduce $\operatorname{rank}(FE) \geqslant k$. As $\operatorname{rank}(FE) \leqslant k$, we have $\operatorname{rank}(FE) = k$ with $FE \in M_k(\mathbf{C})$.

From $FA^nE = (FE)^{n+1}$ we derive $\operatorname{rank}(FA^nE) = k$. This, in turn, implies $\operatorname{rank}(A^n) \geqslant k$, and as $\operatorname{rank}(A^n) \leqslant k$ we conclude $\operatorname{rank}(A^n) = k$ for any $n \geqslant 3$.

PROBLEM 4. Let $f : [0,1] \to [0,1]$ be a continuous and bijective function. Describe the set:

$$A = \{f(x) - f(y) \mid x, y \in [0,1] \setminus \mathbf{Q}\}.$$

(The following result is considered to be known: *there is no one-to-one function between the set of irrational numbers and* $\mathbf{Q}$.)

SOLUTION. It is obvious that $A \subset [-1, 1]$. As f is continuous and one-to-one, we deduce that f is strictly monotonic, that is $\{f(0), f(1)\} = \{0, 1\}$ and $-1, 1 \notin A$. It follows that $A \subset (-1, 1)$. It is easy to see that $0 \in A$ and that $a \in A$ implies $-a \in A$. Suppose, for simplicity that f is strictly increasing (otherwise use $g = 1 - f$). Let $a \in (0, 1)$. Then $a = f(b)$ with $b \in (0, 1)$. For any $x \in (b, 1) \cap (\mathbf{R} \setminus \mathbf{Q}) = I$, there is an unique $y_x \in [0, 1]$, such that $f(x) - f(y_x) = a$. If $y_x \in \mathbf{R} \setminus \mathbf{Q}$, then $a \in A$. In the case $y_x \in \mathbf{Q}$ for any $x \in I$, consider the function $g : I \to \mathbf{Q}$ given by $g(x) = y_x$. The function g is one-to-one from I to $\mathbf{Q}$, a contradiction. It follows that $(-1, 1) \subset A$ concluding that $A = (-1, 1)$.

12th GRADE

PROBLEM 1. Let A be a ring.

a) Show that the set $Z(A) = \{a \in A \mid ax = xa, \text{ for all } x \in A\}$ is a subring of the ring A.

b) Prove that, if any commutative subring of A is a field, then A is a field.

SOLUTION. a) Let $a, b \in Z(A)$. From $(a-b)x = ax - bx = xa - xb = x(a-b)$, and $(ab)x = axb = x(ab)$, $\forall x \in A$, we obtain $a - b \in Z(A)$ and $ab \in Z(A)$. As $1 \in Z(A)$, it follows that $Z(A)$ is a subring.

b) Consider $a \in A - \{0\}$ and $B = Z(A)$. Define $D = \{f(a) \mid f \in B[X]\}$, where $B[X]$ is the ring of polynomials having coefficients in B. It easy to verify that D is a commutative subring of A, that is D is a field. It is obvious that $a^{-1} \in D$. Thus a is invertible and because it was arbitary, we conclude that A itself is a field.

PROBLEM 2. Let $f : [0, 1] \to \mathbf{R}$ be an integrable function such that:

$$0 < \left| \int_0^1 f(x)\mathrm{d}x \right| \leqslant 1.$$

Show that there exist $x_1 \neq x_2$, $x_1, x_2 \in [0, 1]$, such that:

$$\int_{x_1}^{x_2} f(x)\mathrm{d}x = (x_1 - x_2)^{2002}.$$

SOLUTION. Consider $F : [0, 1] \to \mathbf{R}$ defined by $F(x) = \int_0^x f(t)\mathrm{d}t$. We distinguish two cases.

First case: For all $a, b \in [0, 1]$, $a \neq b$, we have $|F(a) - F(b)| \geqslant |a - b|^{2002}$. Then, we can take $x_1 = 1$, $x_2 = 0$ $(1 \geqslant |\int_0^1 f(t)\mathrm{d}t| \geqslant 1^{2002})$.

Second case: For any $x_1 \neq x_2 \in [0,1]$, we have $|F(x_1)-F(x_2)| \leqslant |x_1-x_2|^{2002}$, which in turn can be written

$$\left|\frac{F(x_1)-F(x_2)}{x_1-x_2}\right| \leqslant |x_1-x_2|^{2001}.$$

Letting $x_2 \to x_1$, we derive $F'(x_1)=0$, and as x_1 is arbitrary, we conclude $F'=0$, contradicting thus the condition in the hypothesis $0<|F(1)-F(0)|$.

Therefore, we can find $a,b,c,d \in [0,1]$ such that

$0 \leqslant a < b \leqslant 1$, $|F(b)-F(a)| > |b-a|^{2002}$ and

$0 \leqslant c < d \leqslant 1$, $|F(d)-F(c)| < |d-c|^{2002}$.

Consider the continous function $g(t)=|F(y_t)-F(z_t)|-|y_t-z_t|^{2002}$, where $0 \leqslant z_t=(1-t)a+tb < y_t=(1-t)c+td \leqslant 1$. As $g(0)=|F(d)-F(c)|-|d-c|^{2002}<0$ and $g(1)=|F(a)-F(b)|-|a-b|^{2002}>0$, the intermediate value property implies the existence of $t' \in (0,1)$ such that $g(t')=0$. The points $x_1=y_{t'}$ and $x_2=z_{t'}$ satisfy the conclusion of the problem.

PROBLEM 3. Let $f:\mathbf{R}\to\mathbf{R}$ be a continuous and bounded function such that If

$$x\int_x^{x+1} f(t)\mathrm{d}t=\int_0^x f(t)\mathrm{d}t, \text{ for any } x\in\mathbf{R}.$$

Prove that f is a constant function.

SOLUTION. Consider $F(x)=\int_0^x f(t)\mathrm{d}t$. The given equality can gives $x(F(x+1)-F(x))=F(x)$, that is $\frac{F(x)}{x}=\frac{F(x+1)}{x+1}$, $\forall x\in(0,\infty)$. Define $p(x)=\frac{F(x)}{x}$ as a function $p:(0,\infty)\to\mathbf{R}$. It has finite derivative and p and p' are periodic of period 1. The equality $F(x)=xp(x)$ implies $f(x)=p(x)+xp'(x)$

Suppose that there is $x_0>0$ such that $p'(x_0)\neq 0$. Then $f(x_0+n)=p(x_0+n)+(x_0+n)p'(x_0+n)=p(x_0)+(x_0+n)p'(x_0)$, for all $n\in\mathbf{N}$. It follows that the limit $\lim\limits_{n\to\infty} f(x_0+n)$ is infinite, a contradiction. Thus $p(x)=k$, for all $n\in(0,\infty)$, that is $f(x)=k$, for all $x\in(0,\infty)$.

The case when $x\in(-\infty,0)$ is analoguous. By continuity, $f(x)=k$, for all $x\in\mathbf{R}$.

PROBLEM 4. Let K be a field having $q=p^n$ elements, where p is a prime number and $n\geqslant 2$ is an arbitrary integer number. For any $a\in K$, one defines the polynomial $f_a=X^q-X+a$. Show that:

a) $f=(X^p-X)^q-(X^p-X)$ is divisible by f_1;

b) f_a has at least p^{n-1} essentialy different irreducible factors $K[X]$.

SOLUTION. a) Because $p\cdot 1=0$, one has $(a+b)^p=a^p+b^p$ for any $a,b\in K$ (Frobenius morphism). The divisibility $f_1|f$ is an easy consequence of the formula $(X^p-X)^{p^n}-(X^p-X)=X^{p^{n+1}}-X^{p^n}-X^p+X=(X^{p^n}-X+1)^p-(X^{p^n}-X+1)$.

b) The decomposition of f_0 is $f_0=X^{p^n}-X=\prod\limits_{b\in K}(X-b)$, and it contains p^n irreducible factors.

For any $a\in K^*$, using the fact that $f_a(aX)=a^qX^q-aX+a=af_1(X)$, we infer that it will be sufficient to verify the property for f_1.

Consider the polynomial $g=X^p-X$. From a) and the factorisation formula for f_0 we get $f_1=(f_1,g^q-g)=(f_1,\prod\limits_{b\in K}(g-b))=\prod\limits_{b\in K}(f_1,g-b)$. Given that $\mathrm{gr}f_1=p^n$ and $\mathrm{gr}(f_1,g-b)\leqslant p$, for $b\in K$, we conclude that at least p^{n-1} between the polynomials $(f_1,g-b)$ have the degree $\geqslant 1$. Taking into account that all these polynomials are relatively prime and each of them contains at least an irreducible factor, we get the conclusion.

II.4. ELEMENTARY SCHOOL OLYMPIAD

City of Bucharest

5th GRADE

PROBLEM 1. Show that the number

$$\frac{36}{5\cdot 7}-\frac{1}{5\cdot 6\cdot 7}-\frac{1}{6\cdot 7\cdot 8}-\frac{1}{6\cdot 8}$$

is an integer.

SOLUTION. $\frac{36}{5\cdot 7}-\frac{1}{5\cdot 6\cdot 7}-\frac{1}{6\cdot 7\cdot 8}-\frac{1}{6\cdot 8}=\frac{49}{6\cdot 8}-\frac{1}{6\cdot 8}=1$.

PROBLEM 2. We consider the number

$$N=\frac{1}{10}+\frac{2}{10^2}+\frac{3}{10^3}+\cdots+\frac{11}{10^{11}}.$$

Show that $0.12345679<N<0.1234568$.

SOLUTION. Using decimal representation, one has

$$N=0.12345679011.$$

PROBLEM 3. A sport contests was organized during four days. The medals were distributed as follows: each day half of the existing medals were awarded and one more. How many medals were awarded each of the four days?

SOLUTION. Let t be the number of remaining medals after third day. Then, in the fourth day $\frac{t}{2}+1=t$ medals were distributed. Hence $t=2$.

Let z be the number of remaining medals after second day. Then, in the third day, $\frac{z}{2}+1=z-2$ medals were distributed. Hence $z=6$.

Let y be the number of remaining medals after the first day. In the same way we get $y=14$.

Let x be the total number of medals. Then $\frac{x}{2}+1=x-14$ and $x=30$. So, the number of awarded medals was: 16, 8, 4, 2.

PROBLEM 4. The sets A and B consist each of a finite number of consecutive positive integers. Let a be the arithmetic mean of the elements in A and b be the arithmetic mean of elements in B. The arithmetic mean of a and b is 12 and it is known that $A\cap B=\{12\}$. Find the maximal number of elements in the set $A\cup B$.

SOLUTION. Since $A = \{12\}$ and A, B contain consecutive numbers, we may assume that 12 is the greatest element of A and the least element of B. Since $a \in A$ and $b \in B$, A and B have an odd number of elements. In A we have $12 - a$ elements. From $\frac{a+b}{2} = 12$ one obtains $12 - a = b - 12$. Then A has $2(12 - a) + 1$ elements and B has $2(b - 12) + 1 = 2(12 - a) + 1$ elements. Hence $A \cup B$ has $4(12 - a) + 2 - 1$ elements. The least value for a is 7, hence the greatest N is 21.

6th GRADE

PROBLEM 1. Let $A = \{a \in \mathbf{Z} \mid -2000 \leqslant a \leqslant 2000\}$.
a) Find the sum of elements of A.
b) Show that the sum of absolute values of elements of A is a perfect square.

SOLUTION. The sum of elements in A is 2001.
The sum of absolute values is

$$2(1 + 2 + 3 + \cdots + 2000) + 2001 = 2001^2.$$

PROBLEM 2. Find positive integers a, b which satisfy the conditions:
(i) $6a + b = 330$;
(ii) the least common multiple of a and b is 12 times greater than the greatest common divisor of a and b.

SOLUTION. Let d be the g.c.d. and $a = da', b = db'$. Then $da'b' = 12d$ and $d(6a' + b') = 330$. One gets $a = 45$ and $b = 60$.

PROBLEM 3. Let a, b, c be positive integers such that

$$\frac{a+b}{bc} = \frac{b+c}{ca} = \frac{c+a}{ab}.$$

Show that $a = b = c$.

SOLUTION. The equation is equivalent to

$$a(a + b) = b(b + c) = c(c + d).$$

Let d be the g.c.d. of a, b, c and $a = dA, b = dB, c = dC$. One gets

$$A(A + B) = B(B + C) = C(C + A).$$

If p is prime and a divisor of A, it is necesarily a divisor of B and of C. It follows, as g.c.d$(A, B, C) = 1$, that $A = B = C = 1$ and then $a = b = c$.

PROBLEM 4. Let ABC be an isosceles triangle. The base of triangle ABC is AC, the length of AC is a and $\angle B = 70°$. On the segments AB, AC are given the points D, E respectively, such that $DA + AE = a$. On the segments AC, BC are given the points F, G respectively, such that $FC + CG = a$. The points E, F are distinct. Find the angle between the lines DF and EG.

SOLUTION. Let DF and EG intersect in O. We have $\Delta DAF \equiv \Delta ECG$, since $AF = CG$, $AD = EC$ and $\angle DAF = \angle ECG$. The conclusion is $\angle FOE = 55°$.

Part III: SELECTION EXAMINATIONS
for the International Mathematical Oympiad,
Balkan Mathematical Olympiad
and Junior Balkan Mathematical Olympiad

III.1. PROPOSED PROBLEMS

First selection examination for the 43rd IMO and 19th BMO

Râmnicu Vâlcea, March 21, 2002

PROBLEM 1. Find all pairs of sets A, B, which satisfy the conditions:
(i) $A \cup B = \mathbf{Z}$;
(ii) if $x \in A$, then $x - 1 \in B$;
(iii) if $x \in B$ and $y \in B$, then $x + y \in A$.

Laurenţiu Panaitopol

PROBLEM 2. Let $(a_n)_{n \geqslant 0}$ be the sequence defined as follows: $a_0 = a_1 = 1$ and $a_{n+1} = 14a_n - a_{n-1}$, for any $n \geqslant 1$. Show that the number $2a_n - 1$ is a perfect square, for all positive integers n.

Bogdan Enescu and Titu Andreescu

PROBLEM 3. Let ABC be an acute triangle. The segment MN is the midline of the triangle that is parallel to side BC and P is the projection of the point N on the side BC. Let A_1 be the midpoint of the segment MP. Points B_1 and C_1 are constructed in a similar way. Show that if AA_1, BB_1 and CC_1 are concurent lines, then the triangle ABC has two equal sides.

Mircea Becheanu

PROBLEM 4. For any positive integer n, let $f(n)$ be the number of possible choices of signes $+$ or $-$ in the algebraic expression $\pm 1 \pm 2 \pm \cdots \pm n$, such that the obtained sum is zero. Show that $f(n)$ satisfies the following conditions:
a) $f(n) = 0$ for $n \equiv 1 (\bmod 4)$ or $n \equiv 2 (\bmod 4)$;
b) $2^{\frac{n}{2}-1} \leqslant f(n) < 2^n - 2^{\left[\frac{n}{2}\right]+1}$, for $n \equiv 0 (\bmod 4)$ or $n \equiv 3 (\bmod 4)$.

Ioan Tomescu

Second selection examination for the 43rd IMO and 19th BMO

Bucharest, April 13, 2002

PROBLEM 5. Let $ABCD$ be a unit square. For any interior points M, N, such that the line MN does not contain a vertex of the square, we denote by $s(M, N)$ the least area of triangles having their vertices in the set of points $\{A, B, C, D, M, N\}$. Find the least number k such that $s(M, N) \leqslant k$, for all such points M, N.

Dinu Şerbănescu

PROBLEM 6. Let $P(X)$ and $Q(X)$ be integer polynomials of degree p, q respectively. Assume that $P(X)$ divides $Q(X)$ and all their coefficients are either 1 or 2002. Show that $p + 1$ is a divisor of $q + 1$.

Mihai Cipu

PROBLEM 7. Let a, b be positive real numbers. For any positive integer n, denote by x_n the sum of digits of the number $[an + b]$ in its decimal representation. Show that the sequence $(x_n)_{n\geqslant 1}$ contains a constant subsequence.

Laurenţiu Panaitopol

PROBLEM 8. At an international conference there are four official languages. Any two participants can discuss in one of these languages. Show that at least 60% of the participants can speak the same language.

Mihai Bălună

Third selection examination for the 43rd IMO and 19th BMO

Bucharest, April 14, 2002

PROBLEM 9. Let $ABCDE$ be a cyclic pentagon inscribed in a circle of center O which has angles $\angle B = 120°$, $\angle C = 120°$, $\angle D = 130°$, $\angle E = 100°$. Show that the diagonals BD and CE meet at a point belonging to the diameter AO.

Dinu Şerbănescu

PROBLEM 10. Let $n \geqslant 4$ be an integer and $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$.

Show that the following inequality holds:

$$\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \cdots + \frac{a_n}{a_1^2+1} \geqslant \frac{4}{5}(a_1\sqrt{a_1} + a_2\sqrt{a_2} + \cdots + a_n\sqrt{a_n})^2.$$

Mircea Becheanu and Bogdan Enescu

PROBLEM 11. Let n be a positive integer and S be the set of all positive integers a such that $1 < a < n$ and $a^{a-1} - 1$ is divisible by n. Show that if $S = \{n-1\}$, then n is twice a prime number.

Mihai Cipu and Nicolae Ciprian Bonciocat

PROBLEM 12. Let $f : \mathbf{Z} \to \{1, 2, \ldots, n\}$ be a function that satisfies the condition

- $f(x) \neq f(y)$, for all $x, y \in \mathbf{Z}$ such that $|x - y| \in \{2, 3, 5\}$.

Show that $n \geqslant 4$.

Ioan Tomescu

Fourth selection examination for the 43rd IMO

Bucharest, June 1st, 2002

PROBLEM 13. Let $(a_n)_{n\geqslant 1}$ be a sequence of positive integers defined as follows:

- $a_1 > 0$, $a_2 > 0$;
- a_{n+1} is the least prime divisor of $a_{n-1} + a_n$, for all $n \geqslant 2$.

Show that a real number x whose decimals are the digits of the numbers $a_1, a_2, \ldots, a_n, \ldots$ written in that order, is a rational number (digits are considered in the decimal representation).

Laurenţiu Panaitopol

PROBLEM 14. Find the least positive real number r with the property:

- whatever four disks are considered, each with center in the edges of a unit square and such that the sum of their radii equals r, there exists an equilateral triangle which has its edges in three of these disks.

Radu Gologan

PROBLEM 15. After elections, every parliament member (PM), has his own absolute rating. When the parliament set up, he enters in a group and gets a relative rating, that is the ratio of its own absolute rating to the sum of all absolute ratings of the PM's in the group. A PM can move from a group to another only if in his new group his relative rating is greater. In a given day only one PM can change the group. Show that only a finite number of group move is possible (Remark: a rating is a positive real number).

Kvant

Fifth selection examination for the 43rd IMO

Bucharest, June 2nd, 2002

PROBLEM 16. Let m, n be positive integers of distinct parities and such that $m < n < 5m$. Show that there exist a partition with two element subsets of the set $\{1, 2, 3, \ldots, 4mn\}$ such that the sum of numbers in each set is a perfect square.

Dinu Şerbănescu

PROBLEM 17. Let ABC be a triangle such that $AC \neq BC$, $AB < AC$ and let K be its circumcircle. The tangent line to K at the point A intersects the line BC in the point D. Let K_1 be the circle tangent to K and to the segments (AD), (BD). We denote by M the point where K_1 touches (BD). Show that $AC = MC$ if and only if AM is the bisector line of the angle $\angle DAB$.

Neculai Roman

PROBLEM 18. There are n players, $n \geqslant 2$, which are playing a card game with np cards in p rounds. The cards are coloured in n colours and each colour is labeled with numbers $1, 2, \ldots, p$. The game submits to the following rules:

- each player receives p cards;
- the player who begins the first round throws a card and each player have to discard a card of the same colour, if he has one; otherways they may give an arbitrary card;
- the winner of the round is the player wo has put the greatest card of the same colour as the first one;
- the winner of the round starts the next round with a card that he selects and the play continues with the same rules;
- the played cards are out of the game.

Show that if all cards labelled with number 1 are winners, then $p \geqslant 2n$.

Barbu Berceanu

First selection examination for the 6th Junior Balkan Mathematical Olympiad

Râmnicu Vâlcea, March 21, 2002

PROBLEM 1. For any positive integer n, let

$$f(n) = \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n + 1} + \sqrt{2n - 1}}.$$

Compute the sum $f(1) + f(2) + \cdots + f(40)$.

Titu Andreescu

PROBLEM 2. Let k, n, p be positive integers such that p is a prime number, $k < 1000$ and $\sqrt{k} = n\sqrt{p}$.

a) Prove that if the equation $\sqrt{k + 100x} = (n + x)\sqrt{p}$ has a non-zero integer solution, then p is a divisor of 10.

b) Find the number of all non-negative solutions of the above equation.

Mircea Fianu

PROBLEM 3. Consider a $1 \times n$ rectangle and some tiles of size 1×1 of four different colours. The rectangle is tiled in such a way that no two neighbouring square tiles have the same colour.

a) Find the number of distinct symmetrical tilings.

b) Find the number of tilings such that any consecutive square tiles have distinct colours.

Dan Brânzei

PROBLEM 4. Let $ABCD$ be a parallelogram of center O. Points M and N are the midpoints of BO and CD, respectively. Prove that if the triangles ABC and AMN are similar, then $ABCD$ is a square.

Dinu Şerbănescu

Second selection examination for the 6th Junior Balkan Mathematical Olympiad

Bucharest, April 13, 2002

PROBLEM 5. A square of side 1 is decomposed into 9 equal squares of sides $\frac{1}{3}$ and the one in the center is painted in black. The remaining eight squares are analogously divided into nine squares each, and squares in the centres are painted in black.

Prove that after 1000 steps the total area of the black region exceeds 0.999.

Costel Chiteş and Cristinel Mortici

PROBLEM 6. Find all positive integers a, b, c, d such that

$$a + b + c + d - 3 = ab + cd.$$

Dinu Şerbănescu

PROBLEM 7. Let ABC be an isosceles triangle such that $AB = AC$ and $\angle A = 20°$. Let M be the foot of the altitude from C and let N be a point on the side AC such that $CN = \frac{1}{2}BC$.

Find the measure of the angle $\angle AMN$.

Dinu Şerbănescu

PROBLEM 8. Let $ABCD$ be a unit square. For any interior points M and N such that the line MN does not contain any vertex of the square, denote by $s(M, N)$ the least area of a triangle having vertices in the set $\{A, B, C, D, M, N\}$.

Find the least number k such that $s(M, N) \leqslant k$, for all such points M, N.

Dinu Şerbănescu

Third selection examination for the 6th Junior Balkan Mathematical Olympiad

Bucharest, April 14, 2002

PROBLEM 9. Let n be an even positive integer and let a, b be two relatively prime positive integers.
Find a and b such that $a+b$ is a divisor of a^n+b^n.

Dinu Şerbănescu

PROBLEM 10. The diagonals AC and BD of a convex quadrilateral $ABCD$ meet at O. Let m be the measure of the acute angle formed by these diagonals. A variable angle xOy of measure m intersects the quadrilateral by a convex quadrilateral of constant area.
Prove that $ABCD$ is a square.

Mircea Fianu

PROBLEM 11. A given equilateral triangle of side 10 is divided into 100 equilateral triangles of side 1 by drawing parallel lines to the sides of the original triangle.
Find the number of equilateral triangles, having vertices in the intersection points of that parallel lines and whose sides lie on the parallel lines.

Dinu Şerbănescu

PROBLEM 12. Prove that for any real numbers a, b, c such that $0 < a, b, c < 1$, the following inequality holds

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

Dinu Şerbănescu

Fourth selection examination for the 6th Junior Balkan Mathematical Olympiad

Bucharest, April 15, 2002

PROBLEM 13. Let a be an integer. Prove that for any real number x, $x^2 < 3$, both the numbers $\sqrt{3-x^2}$ and $\sqrt[3]{a-x^3}$ cannot be rational.

Laurenţiu Panaitopol

PROBLEM 14. The last four digits of a perfect square are equal. Prove that all of them are zeros.

Laurenţiu Panaitopol

PROBLEM 15. Let $C_1(O_1)$ and $C_2(O_2)$ be two circles such that C_1 passes through O_2. Point M lies on C_1 such that $M \notin O_1O_2$. The tangents from M at C_2 meet again C_1 at A and B.

Prove that the tangents from A and B at C_2 – others than MA and MB – meet at a point located on C_1.

D. Şerb

PROBLEM 16. Five points are given in the plane such that each of the 10 triangles they define has its area greater than 2. Prove that there exists a triangle of area greater than 3.

Laurenţiu Panaitopol

Fifth selection examination for the 6th Junior Balkan Mathematical Olympiad

Bucharest, April 16, 2002

PROBLEM 17. Let $m, n > 1$ be integer numbers. Solve in positive integers

$$x^n + y^n = 2^m.$$

Laurenţiu Panaitopol

PROBLEM 18. We are given n circles which have the same center. Two lines D_1, D_2 are concurent in P, a point inside all circles. The rays determined by P on the line D_1 meet the circles in points $A_1, A_2, \ldots, A_n$ and $A'_1, A'_2, \ldots, A'_n$ respectively and the rays on D_2 meet the circles at points $B_1, B_2, \ldots, B_n$ and $B'_1, B'_2 \ldots, B'_n$ (points with the same indices lie on the same circle).
Prove that if the arcs A_1B_1 and A_2B_2 are equal then the arcs A_iB_i and $A'_iB'_i$ are equal, for all $i = 1, 2, \ldots n$.

Dinu Şerbănescu

PROBLEM 19. Let ABC be a triangle and $a = BC$, $b = CA$ and $c = AB$ be the lengths of its sides. Points D and E lie in the halfplane determined by BC and A. Suppose that $DB = c$, $CE = b$ and that the area of $DECB$ is maximal. Let F be the midpoint of DE and let $FB = x$.
Prove that $FC = x$ and $4x^3 = (a^2+b^2+c^2)x + abc$.

Dan Brânzei

PROBLEM 20. Let p, q be two distinct primes. Prove that there are positive integers a, b such that the arithmetic mean of all positive divisors of the number $n = p^aq^b$ is an integer.

Laurenţiu Panaitopol

III.2. SELECTION EXAMINATIONS – SOLUTIONS

First IMO-BMO selection examination

PROBLEM 1. Find all pairs of sets A, B, which satisfy the conditions:
(i) $A \cup B = \mathbf{Z}$;
(ii) if $x \in A$, then $x - 1 \in B$;
(iii) if $x \in B$ and $y \in B$, then $x + y \in A$.

SOLUTION. We shall prove that exactly two pairs of sets verify the given conditions: either $A = B = \mathbf{Z}$ or $A = 2\mathbf{Z}, B = 2\mathbf{Z} + 1$.

If $0 \in B$ it follows form (iii) that $x + 0 \in A$, for all $x \in B$. Therefore $B \subset A$ and by (i) we conclude that $A = B = \mathbf{Z}$, in this case.

Suppose now that $O \notin B$. Then by (i) $0 \in A$ and by (ii), $-1 \in B$. Using (ii), we get succesively $-2 = (-1) + (-1) \in A$, $-3 \in B$, $-4 \in A$, and by easy induction $-2n \in A$ and $-2n - 1 \in B$, for all positive integers n.

If $2 \in B$ then $2 + (-1) = 1 \in A$ and $1 - 1 = 0 \in B$, contradiction. Hence $2 \in A$ and $2 - 1 = 1 \in B$.

We shall prove that $2n \in A$ for all n. If by contradiction, $n > 1$ is minimal with $2n \in B$, then $2n - 1 \in A$ by (iii) and $(2n - 1) - 1 = 2(n - 1) \in B$ by (iii), contradicting the minimality condition. Thus, all even positive integers belong to A and do not belong to B, while all odd integers are in B and not in A. As $-1 \in A$ would imply $-2 \in B$, which in turn implies $3 + (-2) = -1 \in B$, we conclude that $-1 \notin A$.

Hence $A = 2\mathbf{Z}$ and $B = 2\mathbf{Z} + 1$ in this case.

PROBLEM 2. Let $(a_n)_{n \geqslant 0}$ be the sequence defined as follows:
$a_0 = a_1 = 1$ and $a_{n+1} = 14a_n - a_{n-1}$, for any $n \geqslant 1$.
Show that the number $2a_n - 1$ is a perfect square, for all positive integers n.

SOLUTION. Consider also the sequence $(b_n)_{n \geqslant 0}$ of integers defined by $b_0 = -1$, $b_1 = 1$ and $b_{n+1} = 4b_n - b_{n-1}$, for all $n \geqslant 1$. We shall prove by induction that $2a_n - 1 = b_n^2$, for all integer n. More precisely, for sake of simplify the induction proof, we shall prove

(i) $2a_n - 1 = b_n^2$;
(ii) $2b_n b_{n-1} = a_n + a_{n-1} - 4$.

It is clear that $b_0^2 = 2a_0 - 1 = 1$, $b_1^2 = 2a_1 - 1$ and $2b_0 b_1 = -2 = a_0 + a_1 - 4$. Assume that equalities (i) and (ii) are true for n. Then

$$\begin{aligned} b_{n+1}^2 &= (4b_n - b_{n-1})^2 = 16b_n^2 - 8b_n b_{n-1} + b_{n-1}^2 \\ &= 16(2a_n - 1) + (2a_{n-1} - 1) - 4(a_n + a_{n-1} - 4) \\ &= 28a_n - 2a_{n-1} - 1 = 2(14a_n - a_{n-1}) - 1 = 2a_{n+1} - 1 \end{aligned}$$

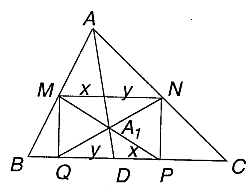
and

$$\begin{aligned} 2b_{n+1} b_n &= 2(4b_n - b_{n-1})b_n = 8b_n^2 - 2b_n b_{n-1} \\ &= 8(2a_n - 1) - a_n - a_{n-1} + 4 = 15a_n - a_{n-1} - 4 \\ &= 14a_n - a_{n-1} + a_n - 4 = a_{n+1} + a_n - 4. \end{aligned}$$

REMARK FOR AN ALTERNATIVE SOLUTION. The terms of the sequence can be computed by using the characteristic equation of the recursive relation. One can prove by induction and using that closed formula, that $2a_n - 1$ is a perfect square.

PROBLEM 3. Let ABC be an acute triangle. The segment MN is the midline of the triangle that is parallel to side BC and P is the projection of the point N on the side BC. Let A_1 be the midpoint of the segment MP. Points B_1 and C_1 are constructed in a similar way. Show that if AA_1, BB_1 and CC_1 are concurent lines, then the triangle ABC has two equal sides.

SOLUTION.



Denote for simplicity the sides BC, CA, AB by a, b, c respectively. Let Q be the projection of the point M on BC. Then A_1 is the centroid of the rectangle $MNPQ$. Let D, E be the intersection points of AA_1 with sides BC and MN resepectively. Denote for simplicity by x, y the lengths of the segments DP, QD respectively. We have $ME = x$, $EN = y$ and $MN = x + y = \frac{a}{2}$.

By similarity, we obtain

$$\frac{BD}{DC} = \frac{ME}{EN} = \frac{x}{y} = \frac{BD + x}{DC + y} = \frac{BP}{CQ} = \frac{\frac{a}{2} + BQ}{\frac{a}{2} + CP} = \frac{a + c\cos B}{a + b\cos C}.$$

In the same way, if we denote by F, G the intersection points of BB_1, CC_1 with CA and AB, respectively, we obtain the ratios

$$\frac{CF}{FA} = \frac{b + a\cos C}{b + c\cos A}, \text{ and } \frac{AG}{GB} = \frac{c + b\cos A}{c + a\cos B}.$$

By Ceva's theorem, the segments AD, BF and CG are concurent if and only if

$$(1) \qquad \frac{a + c\cos B}{a + b\cos C} \cdot \frac{b + a\cos C}{b + c\cos A} \cdot \frac{c + b\cos A}{c + a\cos B} = 1.$$

By cosine's law, we have:

$$a + c\cos B = a + \frac{a^2 + c^2 - b^2}{2a} = \frac{3a^2 + c^2 - b^2}{2a},$$

and the other similar identities. Therefore, (1) becomes

$$\frac{3a^2 + c^2 - b^2}{3a^2 + b^2 - c^2} \cdot \frac{3b^2 + a^2 - c^2}{3b^2 + c^2 - a^2} \cdot \frac{3c^2 + b^2 - a^2}{3c^2 + a^2 - b^2} = 1.$$

Let be for simplicity, $u = c^2 - b^2$, $v = a^2 - c^2$, $w = b^2 - a^2$. The last equality is then equivalent to

$$(3a^2 + u)(3b^2 + v)(3c^2 + w) = (3a^2 - u)(3b^2 - v)(3c^2 - w).$$

Canceling similar members, one obtains

$$18(b^2c^2u + c^2a^2v + a^2B^2w) + 2uvw = 0.$$

Replacing u, v, w by their expresions in a, b, c, we conclude

$$(c^2 - b^2)(b^2 - a^2)(a^2 - c^2) = 0,$$

whence the conclusion of the problem.

PROBLEM 4. For any positive integer n, let $f(n)$ be the number of possible choices of signes + or − in the algebraic expression $\pm 1 \pm 2 \pm \cdots \pm n$, such that the obtained sum is zero. Show that $f(n)$ satisfies the following conditions:

a) $f(n) = 0$ for $n \equiv 1 (\mathrm{mod}\, 4)$ or $n \equiv 2 (\mathrm{mod}\, 4)$;

b) $2^{\frac{n}{2}-1} \leqslant f(n) < 2^n - 2^{[\frac{n}{2}]+1}$, for $n \equiv 0 (\mathrm{mod}\, 4)$ or $n \equiv 3 (\mathrm{mod}\, 4)$.

SOLUTION. The number $f(n)$ can be also defined as the number of partitions of the set $\{1, 2, \ldots, n\}$ in two subsets of equal sums. The sum of elements in each subset being $\frac{1}{2}(1 + 2 + \cdots + n) = \frac{n(n+1)}{4}$, we conclude that this number is an integer if and only if $n \equiv 0$ or $3 \pmod 4$.

Assume that $n = 4k$ or $n = 4k + 3$ for some integer k. Since $1 + 2 - 3 = 0$ and $-1 - 2 + 3 = 0$, one has $f(3) = 2$. In a similar way one shows that $f(4) = 2$.

We shall prove that $f(n + 4) \geqslant 4f(n)$ for all $n \geqslant 3$. Let C_1, C_2 be the two classes of a "good" partition of $\{1, 2, \ldots, n\}$. We can produce four new partitions as follows:

a) $C_1 \cup \{n + 1, n + 4\}$ and $C_2 \cup \{n + 2, n + 3\}$.

b) $C_1 \cup \{n + 2, n + 3\}$ and $C_2 \cup \{n + 1, n + 4\}$.

c) Assume that $1 \in C_2$. Then produce $(C_1 \setminus \{1\}) \cup \{n + 2, n + 4\}$ and $C_2 \cup \{1, n + 1, n + 2\}$.

d) Assume that $2 \in C_1$. Then we produce $(C_1 \setminus \{2\}) \cup \{n + 3, n + 4\}$ and $C_2 \cup \{2, n + 1, n + 2\}$.

This argument shows that for $n + 4$ we have at least four times more "good" partitions, that is $f(n + 4) \geqslant 4f(n)$, for any $n \geqslant 3$.

Finally, if $n = 4k$, we obtain

$$f(4k) \geqslant 4f(4k-4) \geqslant 4^2 f(4k-4\cdot 2) \geqslant \cdots \geqslant 4^{k-1} f(4k-4(k-1)) = 2\cdot 4^{k-1} = 2^{\frac{n}{2}-1},$$

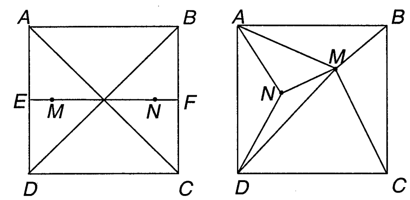
and for $n = 4k + 3$, we get

$$f(4k + 3) \geqslant 4f(4(k - 1) + 3) \geqslant \cdots \geqslant 4^k f(3) = 2^{2k+1} = 2^{\frac{n}{2}-\frac{1}{2}} > 2^{\frac{n}{2}-1}.$$

Second IMO-BMO selection examination

PROBLEM 5. Let $ABCD$ be a unit square. For any interior points M, N, such that the line MN does not contain a vertex of the square, we denote by $s(M, N)$ the least area of triangles having their vertices in the set of points $\{A, B, C, D, M, N\}$. Find the least number k such that $s(M, N) \leqslant k$, for all such points M, N.

SOLUTION. Let E, F be the midpoints of AD and BC respectively and let M, N be the midpoints of OE and OF. It is easy to check that $s(M, N) = \frac{1}{8}$, that is $k \geqslant \frac{1}{8}$. We shall prove that $k = \frac{1}{8}$.



Notice that $\mathrm{Area}(BMC) + \mathrm{Area}(AMD) = \frac{1}{2}$, for any interior point M. We may assume that N is an interior point of the triangle AMD. Therefore

$$\mathrm{Area}(AND) + \mathrm{Area}(ANM) + \mathrm{Area}(DNM) + \mathrm{Area}(BMC) = \frac{1}{2}.$$

It follows that one of the triangle involved in the preceeding sum has area greater than $\frac{1}{8}$.

AUTHOR NOTE. A similar problem for a single interior point leads to the same answer: $k = \frac{1}{8}$. The same question for three points is a more challenging problem.

PROBLEM 6. Let $P(X)$ and $Q(X)$ be integer polynomials of degree p, q respectively. Assume that $P(X)$ divides $Q(X)$ and all their coefficients are either 1 or 2002. Show that $p + 1$ is a divisor of $q + 1$.

SOLUTION. Let $p(X)$ and $q(X)$ be the polynomials obtained after considering modulo 3 the coefficients of $P(X)$ and $Q(X)$, respectively. It follows that $p(X)$ is a divisor of $q(X)$ in the ring $\mathbf{Z}_3[X]$. Since $p(X) = X^p + X^{p-1} + \cdots + X + 1$ and $q(X) = X^q + X^{q-1} + \cdots + X + 1$, it follows that $X^{p+1} - 1 | X^{q+1} - 1$ in $\mathbf{Z}_3[X]$. By standard considerations we get $\mathrm{g.c.d}(X^{p+1} - 1, X^{q+1} - 1) = X^d - 1$, where $d = \mathrm{g.c.d.}(p + 1, q + 1)$. It follows that $d = p + 1$, and $p + 1 | q + 1$.

PROBLEM 7. Let a, b be positive real numbers. For any positive integer n, denote by x_n the sum of digits of the number $[an + b]$ in its decimal representation. Show that the sequence $(x_n)_{n \geqslant 1}$ contains a constant subsequence.

SOLUTION. For any integer k we denote $n_k = \left[\frac{10^k + a - b}{a}\right]$. Then

$$10^k = a\left(\frac{10^k + a - b}{a} - 1\right) + b < an_k + b = a\left[\frac{10^k + a - b}{a}\right] + b \leqslant 10^k + b.$$

It follows that

$$10^k = [an_k + b] \leqslant 10^k + [b].$$

If k is sufficiently large, that is $10^{k-1} > b$, it follows from above that x_{n_k} is 1 plus the sum of the digits of one of the numbers t in the set $\{0, 1, \dots, [b]\}$. Since k takes infinitely many values and the set of numbers t is finite, it follows that for infinitely many k, the sum of digits of numgers $[an_k + b]$ is the same.

PROBLEM 8. At an international conference there are four official languages. Any two participants can discuss in one of these languages. Show that at least 60% of the participants can speak the same language.

SOLUTION. Denote, for simplicity, by $1, 2, 3, 4$ the spoken languages. We distinguish the following three cases.

1. There is a participant who speakes only one language. It is then clear that all participants have to speak it.

2. Any participant is speaking at least two of the given languages but there is no language spoken by all those who speak exactly two. In that case, suppose by symmetry, that the groups of spoken languages that participants know are $(12), (13), (23), (123), (124), (134), (234)$ and (1234). If languages $1, 2, 3$ are not spoken each, by at least $\frac{3}{5}$ of the participants, then, denoting by $x_{ij\dots}$ the number of those who speak simultaneously the languages $i, j, \dots$, we get

$$x_{12} + x_{13} + x_{123} + x_{124} + x_{134} + x_{1234} < \frac{3}{5} \sum_{i,j\dots} x_{ij\dots}$$

$$x_{12} + x_{23} + x_{123} + x_{124} + x_{234} + x_{1234} < \frac{3}{5} \sum_{i,j\dots} x_{ij\dots}$$

$$x_{13} + x_{23} + x_{123} + x_{134} + x_{234} + x_{1234} < \frac{3}{5} \sum_{i,j\dots} x_{ij\dots}.$$

Summing up, we get

$$2 \sum x_{ij\dots} + x_{123} + x_{1234} < \frac{9}{5} \sum x_{ij\dots},$$

which is a contradiction.

3. Any participant speakes at least two languages and there is a language that is common to those speaking exactly two languages. Suppose that this case is described by the following groups of languages $(12), (13), (14), (123), (124), (134), (234), (1234)$. Assume be contradiction the required condition on each of the languages 1,2,3,4, and obtain:

$$x_{12} + x_{13} + x_{14} + x_{123} + x_{124} + x_{134} + x_{1234} < \frac{3}{5} \sum_{i,j\dots} x_{ij\dots}$$

$$x_{12} + x_{123} + x_{124} + x_{234} + x_{1234} < \frac{3}{5} \sum_{i,j\dots} x_{ij\dots}$$

$$x_{13} + x_{123} + x_{134} + x_{234} + x_{1234} < \frac{3}{5} \sum_{i,j\dots} x_{ij\dots}$$

$$x_{14} + x_{124} + x_{134} + x_{224} + x_{1234} < \frac{3}{5} \sum_{i,j\dots} x_{ij\dots}.$$

Multiplying the first inequality by 2 and summing up, we get:

$$3(x_{12} + x_{13} + x_{14}) + 4(x_{123} + x_{124} + x_{134}) + 3x_{234} + 5x_{1234} < 3 \sum x_{ij\dots}.$$

This is a contradiction.

ALTERNATIVE SOLUTION. Let n be the number of persons attending the conference. Consider the incidence matrix (a_{ij}), given by

$$a_{ij} = 1 \text{ if and only if language } i \text{ is spoken by person } i,$$

for any $i = 1, 2, 3, 4$ and $j = 1, 2, \dots n$.

We have to prove that there is a row having more than 60% of ones as its entries. Suppose that the last row contains the greatest number of ones and arrange the other rows in the succesion of the number of ones. We get the following block matrix.

1	0	1	1	1	1	1
1		0	1	1	0	1
1		0	0	1	1	0
1		1	1	0	0	0
a		x		y		z

We distinguish the cases:

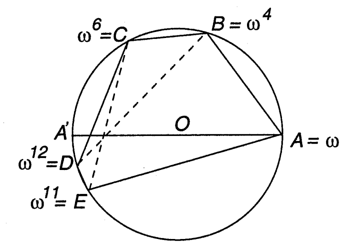
1. $a < \frac{2n}{5}$. It easily implies $x + y + z \geqslant \frac{3n}{5}$.

2. $a \geqslant \frac{2n}{5}$. We can suppose $x < \frac{n}{5}$ and $y < \frac{n}{5}$, otherwise $a + x$ or $a + y$ will satisfy the conclusion. We conclude $x + y < \frac{2n}{5}$ and $z \geqslant \frac{n}{5}$ which imply $a + z \geqslant \frac{3n}{5}$.

Second IMO-BMO selection examination

PROBLEM 9. Let $ABCDE$ be a cyclic pentagon inscribed in a circle of center O which has angles $\angle B = 120°$, $\angle C = 120°$, $\angle D = 130°$, $\angle E = 100°$. Show that the diagonals BD and CE meet at a point belonging to the diameter AO.

SOLUTION. We shall use complex numbers. By standard computations, we find that, on the circumscribed circle, the sides of the pentagon are supported by the following arcs: $\text{arc}AB = 80°$, $\text{arc}BC = 40°$, $\text{arc}CD = 80°$, $\text{arc}DE = 20°$ and $\text{arc}EA = 140°$. It is then natural to consider all these measures as multiples of $20°$ which corresponds to the 18^{th}-primitive root of unity, say $\omega = \cos\frac{2\pi}{18} + \mathrm{i}\sin\frac{2\pi}{18}$. We thus asign, to each vertex, starting from $A = 1$ the corresponding root of unity: $B = \omega^4$, $C = \omega^6$, $D = \omega^{10}$, $E = \omega^{11}$. We shall use the following properties of ω: $\omega^{18} = 1$, $\omega^9 = -1$, $\bar{\omega}^k = \omega^{18-k}$ and $\omega^6 - \omega^3 + 1 = 0$.



We need to prove that the affix of the common point of the lines BD and CE is a real number.

The equation of the line BD is

$$(1)\qquad \begin{vmatrix} z & \bar{z} & 1 \\ \omega^4 & \bar{\omega}^4 & 1 \\ \omega^{10} & \bar{\omega}^{10} & 1 \end{vmatrix} = 0,$$

and the equation of the line CE is

$$(2)\qquad \begin{vmatrix} z & \bar{z} & 1 \\ \omega^6 & \bar{\omega}^6 & 1 \\ \omega^{11} & \bar{\omega}^{11} & 1 \end{vmatrix} = 0.$$

The equation (1) can be written as follows:

$$z(\omega^{14} - \omega^8) - \bar{z}(\omega^4 - \omega^{10}) + (\omega^{12} - \omega^6) = 0,$$

or

$$z\omega^8(\omega^6 - 1) + \bar{z}\omega^4(\omega^6 - 1) + \omega^6(\omega^6 - 1) = 0.$$

Using the properties of ω we derive the simplified version of (1):

$$(1')\qquad z\omega^4 + \bar{z} + \omega^2 = 0.$$

In the same way, equation (2) becomes

$$(2')\qquad z\omega + \bar{z} - \omega^3(\omega^4 - 1) = 0.$$

From (1′) and (2′) we obtain the following expression for z

$$z = \frac{-\omega^7 + \omega^3 - \omega^2}{\omega^4 - \omega} = \frac{-\omega^6 + \omega^2 - \omega}{\omega^6} = -1 + \frac{\omega - 1}{\omega^5}.$$

To prove that z is real, it will suffice to prove that it coincides with its conjugate. It is easy to see that

$$\frac{\omega - 1}{\omega^5} = \frac{\bar{\omega} - 1}{\bar{\omega}^5}$$

is equivalent to

$$\bar{\omega}^4 - \bar{\omega}^5 = \omega^4 - \omega^5,$$

that is $\omega^{14} - \omega^{13} = \omega^4 - \omega^5$, which is true due to the relations verified by ω.

PROBLEM 10. Let $n \geqslant 4$ be an integer and $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$. Show that the following inequality holds:

$$\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \cdots + \frac{a_n}{a_1^2+1} \geqslant \frac{4}{5}(a_1\sqrt{a_1} + a_2\sqrt{a_2} + \cdots + a_n\sqrt{a_n})^2.$$

SOLUTION. By the Cauchy-Schwarz inequality, we get

$$((\sqrt{x_1})^2 + \cdots + (\sqrt{x_n})^2)\left(\left(\frac{a_1}{\sqrt{x_1}}\right)^2 + \cdots + \left(\frac{a_n}{\sqrt{x_n}}\right)^2\right) \geqslant (a_1 + a_2 + \cdots + a_n)^2,$$

for any positive $x_1, x_2, \ldots, x_n$, that is

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \cdots + \frac{a_n^2}{x_n} \geqslant \frac{(a_1 + a_2 + \cdots + a_n)^2}{x_1 + x_2 + \cdots + a_n}.$$

When applying this inequality to the left member of the given inequality, one gets

$$\frac{a_1}{a_2^2+1} + \frac{a_2}{a_3^2+1} + \cdots + \frac{a_n}{a_1^2+1} = \frac{a_1^3}{a_1^2a_2^2 + a_1^2} + \frac{a_2^2}{a_2^2a_3^2 + a_2^2} + \cdots + \frac{a_n^2}{a_n^2a_1^2 + a_n^2}$$
$$\geqslant \frac{(a_1\sqrt{a_1} + a_2\sqrt{a_2} + \cdots + a_n\sqrt{a_n})^2}{a_1^2a_2^2 + a_2^2a_3^2 + \cdots + a_n^2a_1^2 + 1}.$$

Therefore, it will suffice to prove the following inequality

$$a_1^2a_2^2 + a_2^2a_3^2 + \cdots + a_n^2a_1^2 \leqslant \frac{1}{4},$$

where $n \geqslant 4$ and $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$.

This inequality is general: if $n \geqslant 4$ and positive numbers $x_1, x_2, \ldots x_n$ are given, such that $x_1 + x_2 + \cdots + x_n = 1$, then

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1 \leqslant \frac{1}{4}.$$

If n is even, the proof is immediate, as

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1 \leqslant (x_1 + x_3 + \cdots)(x_2 + x_4 + \cdots x_n) \leqslant \frac{1}{4},$$

since the product of two positive numbers of sum 1 has $\frac{1}{4}$ as its maximum.

If n is odd, $n \geqslant 5$, we may consider $x_1 \geqslant x_2$, and because

$$x_1x_2 + x_2x_3 + x_3x_4 \leqslant x_1(x_2 + x_3) + (x_2 + x_3)x_4,$$

we may replace the numbers $x_1, x_2, \ldots, x_n$ by the $n - 1$ numbers $x_1, x_2 + x_3, x_4, \ldots, x_n$. The left member of the given inequality increases, the sum remains one, and we have an even set of numbers, so the preceeding argument can be applied.

PROBLEM 11. Let n be a positive integer and S be the set of all positive integers a such that $1 < a < n$ and $a^{a-1} - 1$ is divisible by n. Show that if $S = \{n - 1\}$, then n is twice a prime number.

SOLUTION. We prove first that n is square-free. Assume that $n = q^b r$, where q is a prime divisor of n, $b \geqslant 2$ and r is a positive integer coprime to q. If $r \geqslant 2$, the number $a = \varphi(q^b)r + 1 = (q^b - q^{b-1})r + 1$ has the property $a \equiv 1 \bmod r$, implying that $a^{a-1} \equiv 1$ modulo r. When $b \geqslant 2$, then q is a divisor of $\varphi(q^b)$, and hence g.c.d$(q, a) = 1$. By Euler's theorem, we obtain $a^{\varphi(q^b)} \equiv 1(\bmod q^b)$, and therefore $a^{a-1} = a^{\varphi(q^b)r} \equiv 1(\bmod q^b)$. By the Chinese remainders theorem, we obtain $a^{a-1} \equiv 1(\bmod n)$, and since $\varphi(q^b)r + 1 < q^b r$, we obtain $1 < q^{b-1}r$. That is a contradiction.

We may thus assume that $n = 2qr$, where $q > 2$ is a prime number and $r > 2$ is an odd number, relatively prime to q. In the case q is not divisible by $r - 1$, it follows that the number $a = (q-1)r+1$ satisfies $a \equiv 1(\bmod 2r)$ and g.c.q.$(a, q) = 1$. By Fermat's theorem, we obtain $a^{q-1} \equiv 1(\bmod q)$ and $a^{a-1} = 1(\bmod q)$. We thus get a solution of the congruence $a^{a-1} \equiv 1(\bmod n)$ with $1 < a < n$.

The last case is when $n = 2pqr$, where p, q are distinct odd primes, $r \geqslant 1$, $p|(qr - 1)$ and $q|(pr - 1)$. We show that the number $a = (p - 1)(q - 1)r + 1$ has the properties in the hypothesis of the problem. Indeed, as p is not a divisor of $(q-1)r-1$ nor q is a divisor of $(p-1)r-1$, a is relatively prime to pq. By Fermat's theorem and the Chinese remainders theorem, we get as in the first part that a satisfies: $a^{a-1} \equiv 1(\bmod n)$ and $1 < a < n$.

It remains only the possibility $n = 2p$, with p prime.

PROBLEM 12. Let $f : \mathbf{Z} \to \{1, 2, \ldots, n\}$ be a function that satisfies the condition

- $f(x) \neq f(y)$, for all $x, y \in \mathbf{Z}$ such that $|x - y| \in \{2, 3, 5\}$. Show that $n \geqslant 4$.

SOLUTION. We shall identify a function $f : \mathbf{Z} \to \{1, 2, \ldots, n\}$ with a bi-infinite sequence $(x_k)_{k \in \mathbf{Z}}$, where $x_k \in \{1, 2, \ldots, n\}$. We have to prove that if such a sequence has the property that it does not contain the same values at distance 2,3 or 5 apart, then $n \geqslant 4$.

It is clear that such a sequence does not exist if $n = 1$. For $n = 2$ it is clear that any block of five consecutive terms contains two having the same value at distance 2 or 3.

For $n = 3$ take a minimal block of $s > 2$ consecutive terms such that the first and the last term are equal and all terms inside the block are different from those at the extremities. It is clear that $s < 7$. For if not, the block obtained by removing the extremities is formed only by two values, and thus discussed in the preceeding case. For $s = 6$ we contradict the condition of distance 5. For $s = 5$ the generic case is a, b, b, c, a followed by b or c. All possibilities give a contradiction. For $s < 5$ the analysis is obvious.

The sequence of blocks $\ldots 1, 2, 3, 4, 1, 2, 3, 4, 1 \ldots$ satisfies the condition for $n = 4$.

Fourth IMO-BMO selection examination

PROBLEM 13. Let $(a_n)_{n \geqslant 1}$ be a sequence of positive integers defined as follows:

- $a_1 > 0$, $a_2 > 0$;

- a_{n+1} is the least prime divisor of $a_{n-1} + a_n$, for all $n \geqslant 2$.

Show that a real number x whose decimals are the digits of the numbers $a_1, a_2, \ldots, a_n, \ldots$ written in that order, is a rational number (digits are considered in the decimal representation).

SOLUTION. It is easy to see that there is a 2 among the first five members of the sequence. This can be shown by considering the parity of the members.

If $a_i = a_{i+1} = 2$, then $a_i = 2$ for all $n \geqslant i$.

If one obtains one of the pairs $2, 3$ or $3, 2$, then the sequence is periodic; for example

$$2, 3, 5, 2, 7, 3, 2, 5, 7, 2, 3, \ldots$$

Let us assume $a_i = 2$ and a_{i+1} is an odd prime. Then we have either $a_{i+1} \equiv 1(\bmod 6)$ or $a_{i+1} \equiv -1(\bmod 6)$. The pairing $2, 6k + 1$ gives rise to the sequence

$$2, 6k + 1, 3, 2, \ldots$$

and one obtains a periodic sequence.

We have to check only the pairing $a_i = 2$, $a_{i+1} = 6k - 1$. Then $a_{i+1}|6k + 1$. If $a_{i+2} \equiv 1(\bmod 6)$ we obtain the sequence

$$2, 6k - 1, 6l + 1, 2, 3, \ldots$$

which becames periodic.

Assume that $a_{i+2} \equiv -1(\bmod 6)$. Since $a_{i+2} = 6l - 1$, the inequalities

$$6l - 1 < \frac{6k + 1}{2} < 6k - 1,$$

imply $l < k$ and $a_{i+2} < a_{i+1}$. Moreover, $a_{i+3} = 2$.

When assuming that $a_{i+4} \equiv -1(\bmod 6)$, one obtains $a_{i+4} < a_{i+2} < a_{i+1}$.

Repeating the above procedure, with the assumption that all odd primes from the sequence are congruent to -1 modulo 6, we obtain a decreasing sequence of prime numbers. It cannot be infinite, thus there exists an i such that $a_i = 2$ and $a_{i+1} = 6k + 1$.

In conclusion, all possibilities give rise to a sequence $a_1, a_2, \ldots, a_n, \ldots$, that becomes periodic. Thus x is a rational number.

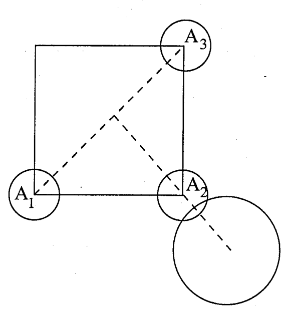
PROBLEM 14. Find the least positive real number r with the property:

- whatever four disks are considered, each with center in the edges of a unit square and the sum of their radii equals r, there exists an equilateral triangle which has its edges in three of these disks.

SOLUTION. Let us denote by $D(A, x)$ the disk of center A and radius x, the border included. We shall use the following lemma.

LEMMA. Let $D_1 = D(O_1, r_1)$ and $D_2 = D(O_2, r_2)$ be two disks and O_3 be the third vertex of the equilateral triangle $O_1O_2O_3$. Let ABC be an equilateral triangle with variable vertices B, C, in the disks D_1, D_2 respectively. Then, the locus of the vertex A is the disk $D_3(O_3, r_1+r_2)$, and also its reflection with respect to the line O_1O_2.

PROOF OF THE LEMMA. The vertices of the equilateral triangles O_1CA when $C \in D_2$ can be obtained by rotating the disk D_2 about O_2 with an angle of $\pm 60°$. When B_1 is arbitrary, one may add a translaton with vector of length at most r_1. Since the triangles are equilateral, it is clear that any point D_3 is obtained in this way.



For the solution of the problem, let $A_1A_2A_3A_4$ be the unit square and $D_i = D(A_i, r_i)$, $i = 1,2,3,4$, be the disks and $r = \sum r_i$. We may assume without loss of generality that $r_1 + r_2 + r_3 \geqslant \frac{3}{4}r$, this being the best evaluation for a minimal sum of three radii.

Consider the disk $D(A, r_1 + r_3)$, by using the disks D_1, D_3 (see figure). It intersects D_2 if and only if

$$r_1 + r_2 + r_2 \geqslant \sqrt{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}.$$

We conclude that the minimal required value is $r = \frac{2(\sqrt{6}-\sqrt{2})}{3}$.

It can be seen that the same value is obtained when two consecutive vertices are considered.

PROBLEM 15. After elections, every parliament member (PM), has his own absolute rating. When the parliament set up, he enters in a group and gets a relative rating. The relative rating is the ratio of its own absolute rating to the sum of all absolute ratings of the PM's in the group. A PM can move from a group to another only if in his new group his relative rating is greater. In a given day, only one PM can change the group. Show that only a finite number of group moves is possible (Remark: a rating is a positive real number).

SOLUTION. Suppose that an arbitrary MP has rating R. When he enters a group, say group G_i, he carries the relative rating $r_i = \frac{R}{S_i}$, where $S_i = \sum_{MP \in G_i} R$. When he moves from the group G_i to the group G_j, the given condition translates into the inequality

$$\frac{R}{S_i} < \frac{R}{S_j + R}.$$

It is obviously equivalent to $R + S_j - S_i < 0$.

After moving from G_i to G_j, one has new sums of absolute ratings:

$$S_i' = S_i - R, \quad S_j' = S_j + R.$$

The idea is to associate to the Parliament a daily invariant:

$$L = \sum_i S_i^2,$$

the sum being considered after all groups G_i. When a MP moves from a group to another, the invariant L changes value to a new one

$$L' = \sum_{k \neq i,j} S_k + (S_i - R)^2 + (S_j + R)^2.$$

It is easy to see that

$$L' - L = (S_i - R)^2 + (S_j + R)^2 - S_i^2 - S_j^2 = 2R(R + S_j - S_i) < 0.$$

Therefore $L' < L$. Since there are only finitely many possibilities to arrange the PM's in groups, the invariant L can take only a finite number of values. Thus it cannot decrease indefinitely.

Fifth IMO selection examination

PROBLEM 16. Let m, n be positive integers of distinct parities and such that $m < n < 5m$. Show that there exist a partition with two element subsets of the set $\{1, 2, 3, \ldots, 4mn\}$ such that the sum of numbers in each set is a perfect square.

SOLUTION. We look for a, b, $a < b$, such that
(1) $(2a+1)^2 + 4mn = (2b+1)^2$, and
(2) $(2a+1)^2 < 4mn$.
We group consecutive numbers $1, 2, \ldots, 4mn$, in the following way

$$\underbrace{\underbrace{1, 2, \ldots, (2a+1)^2 - 2}, (2a+1)^2 - 1}, \underbrace{(2a+1)^2, \underbrace{(2a+1)^2, \ldots, 4mn-1}, 4mn}.$$

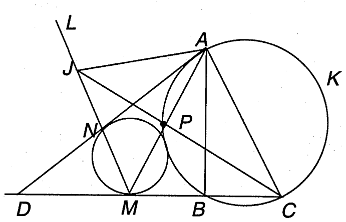
By (1), we have $mn = (b-a)(b+a+1)$. Since $m < n$, put $m = b - a, n = b + a + 1$, and obtain

$$a = \frac{n - m - 1}{2}, \quad b = \frac{m + n - 1}{2}.$$

They are positive integers. As for (2), since $2a+1 = n-m$, we get $(m-n)^2 < 4mn$. The former inequality is equivalent to $m^2 - 6mn + n^2 < 0$ or $\frac{m}{n} < 3 + 2\sqrt{2}$. This is true because $m < 5n$.

PROBLEM 17. let ABC be a triangle such that $AC \neq BC$, $AB < AC$ and let K be its circumcircle. The tangent line to K at the point A intersects the line BC in the point D. Let K_1 be the circle tangent to K and to the segments (AD), (BD). We denote by M the point where K_1 touches (BD). Show that $AC = MC$ if and only if AM is the bisector line of the angle $\angle DAB$.

SOLUTION. Let N be the tangent point of K_1 and AD and J be the center of the excribed circle which is tangent to the side AB. For convenience, we will denote by A, B, C the measures of the angles of the triangle ABC, respectively. Then $\angle JAC = 90° + \frac{1}{2}A$, $\angle BAD = C$ and $\angle ADC = B - C$. It follows that $\angle DMN = \frac{1}{2}(180° - B + c)$.



We shall use the following preliminary result.

LEMMA. The points J, M, N are colinear. E

PROOF. Let L be the intersection point of the lines MN and AC. By Casey's theorem, we have $aNA + bMB = cMC$, and also

$$a\frac{NA}{MC} + b\frac{MB}{MC} = c. \tag{1}$$

By Menelaos theorem applied to the line ML and triangle ADC, we have

$$\frac{LA}{LC} \cdot \frac{MC}{MD} \cdot \frac{ND}{NA} = 1$$

and then $\frac{LA}{LC} = \frac{NA}{MC}$. From (1) and the last relation, weobtain

$$a\frac{LA}{LC} + b\frac{MB}{MC} = c.$$

It follows that MN passes through the center of K_1.

From the lemma and the formula for $\angle DMN$ we get:

$$\angle JMC = B + \frac{1}{2}A. \tag{2}$$

Let us assume $AC = MC$; thus we have $\angle AMC = \angle MAC$ and CJ is the perpendicular bisector of the segment AM. The triangles JAC and JMC are congruent and then $\angle JAC = \angle JMC$. Since $\angle JAC = 90° + \frac{1}{2}A$, taking into account also (2), we get $90° + \frac{1}{2}A = B + \frac{1}{2}A$, that is $B = 90°$.

Hence $\angle CAD = 90°$ too. Since $B = \angle CAD$, we have

$$\angle AMC + \angle MAB = \angle MAC + \angle MAD,$$

and thus $\angle MAD = \angle MAB$. So, AM is the bisector line of $\angle DAB$.

Let us assume that $\angle MAB = \angle MAD$. Let P be the intersection point of AD and K. Because $\angle MAB = \angle MAD$, it follows that the arcs AP and BP on the circle K are equal, AP is the bisector of $\angle ACD$ and AP passes through J. Moreover $PA = PB$.

Again for angles $\angle JBP = \angle JBA - \angle PBA$ and

$$\angle JBP = \frac{A+C}{2} - \frac{C}{2} = \frac{A}{2}. \tag{3}$$

In the triangle JBC, $\angle BJB = 180° - \frac{1}{2}C - B - \frac{A+C}{2} = \frac{1}{2}A$. From this estimation and (3) one gets $PJ = PB$. Moreover $\angle JMP = B + \frac{1}{2}A - B + \frac{1}{2}C$ and $\angle JMP = \frac{A+C}{2}$. In the triangle JMC we have $\angle MJC = 180° - \frac{C}{2} - B - \frac{A}{2}$, that is $\angle MJC = \frac{A+C}{2}$. Hence $PJ = PM$. Since we have also $PA = PB = PM$ it follows that in ΔAMC the segment CP is median and bisector of the angle C. One obtains $AC = MC$.

PROBLEM 18. There are n players, $n \geqslant 2$, which are playing a card game with np cards in p rounds. The cards are coloured in n colours and each colour is labeled with numbers $1, 2, \ldots, p$. The game submits to the following rules:

- each player receives p cards.
- the player who begins the first round throws a card and each player have to discard a card of the same colour, if he has one; otherways they can give an arbitrary card.
- the winner of the round is the player wo has put the greatest card of the same colour as the first one.
- the winner of the round starts the next round with a card that he selects and the play continues with the same rules.
- the played cards are out of the game.

Show that if all cards labelled with number 1 are winners, then $p \geqslant 2n$.

SOLUTION. Let $t_1, t_2, \ldots, t_n$ be the number of rounds in which the cards labeled by 1 win the round. We may assume that these labeled 1 cards have succesively the colours $c_1, c_2, \ldots, c_n$ and players $J_1, J_2 \ldots, J_n$ have played them. The solution is described in the following steps.

Step 1. In the round t_i, all cards of colour c_i which are still unused, belong to the player J_i (since the others don't have this colour).

Step 2. In the round t_n, any colour belongs to at most one player (use step 1).

Step 3. The players $J_1, J_2, \ldots, J_n$ are distinct. It follows from step 2 and the equal numbers of players and colours.

Step 4. We have $t_1 \geqslant 2$. If assuming the contrary, in the first round the card labeled 1 of colour c_1 is winning and J_1 has all cards of colour c_1. It follows that J_1 wins the temaning rounds by using only cards of colour c_1.

Step 5. $t_{i+1} \geqslant t_i + 2$. The player J_i wins the round J_i by playing card 1 of colour c_i. It follows that the card labeled 1 of colour c_{i+1} belongs to a player J_{i+1} which is distinct from J_i.

From steps 4 and 5 it follows that $p \geqslant t_n \geqslant 2n$. The player J_{i+1} should play an intermediate round between rounds t_i and t_{i+1}.

First JBMO selection examination

PROBLEM 1. For any positive integer n, let

$$f(n) = \frac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n+1} + \sqrt{2n-1}}.$$

Compute the sum $f(1) + f(2) + \cdots + f(40)$.

SOLUTION. Let $a = \sqrt{2n+1}$ and $b = \sqrt{2n-1}$. Then $a^2 + b^2 = 4n$, $ab = 4n^2 - 1$ and $a^2 - b^2 = 2$. Hence

$$f(n) = \frac{a^2 + b^2 + ab}{a + b} = \frac{a^3 - b^3}{a^2 - b^2} = \frac{1}{2}\left(\sqrt{(2n+1)^3} - \sqrt{(2n-1)^3}\right)$$

and thus

$$f(1) + f(2) + \cdots + f(40) = \frac{1}{2}(\sqrt{3^3} - \sqrt{1^3} + \sqrt{5^3} - \sqrt{3^3} + \cdots + \sqrt{81^3} - \sqrt{79^3})$$
$$= \frac{1}{2}(\sqrt{81^3} - \sqrt{1^3}) = \frac{1}{2}(9^3 - 1) = \frac{1}{2}(729 - 1) = 364.$$

PROBLEM 2. Let k, n, p be positive integers such that p is a prime number, $k < 1000$ and $\sqrt{k} = n\sqrt{p}$.

a) Prove that if the equation $\sqrt{k + 100x} = (n + x)\sqrt{p}$ has a non-zero integer solution, then p is a divisor of 10.

b) Find the number of all non-negative solutions of the above equation.

SOLUTION. a) By squaring the members of the equation we get $k + 100x = n^2p + 2nxp + x^2p$, or $100 = p(2np + x)$. The conclusion follows from the fact that p is a prime number.

b) If $p = 2$, then $50 = 2n + x$ and $0 \leqslant n \leqslant 25$. Since $n^2 = \frac{k}{p} = \frac{k}{2} < 500$, it follows that $n \leqslant 22$ and we have 23 solutions.

If $p = 5$, then $20 = 2n + x$ and $0 \leqslant n \leqslant 10$. Notice that $n^2 = \frac{k}{p} = \frac{k}{5} < 200$ for any $n \leqslant 10$, therefore we have other 11 solutions. We have 34 solutions in all.

PROBLEM 3. Considet a $1 \times n$ rectangle and some tiles of size 1×1 of four different colours. The rectangle is tiled in such a way that no two neighboring square tiles have the same colour.

a) Find the number of distinct symmetrical tilings.

b) Find the number of tilings such that any consecutive square tiles have distinct colours.

SOLUTION. a) If $n = 2k$, there are no such symmetrical tilings (otherwise the k and $k + 1$ tiles must have the same colour).

If $n = 2k + 1$, the problem is to count the possible tilings for $k + 1$ squares. There are $4 \cdot \underbrace{3 \cdot 3 \cdots 3} = 4 \cdot 3^k$ such tilings.

b) There are $4 \cdot 3 \cdot \underbrace{2 \cdot 2 \cdots 2} = 4 \cdot 3 \cdot 2^{n-2}$ tilings.

PROBLEM 4. Let $ABCD$ be a parallelogram of center O. Points M and N are the midpoints of BO and CD, respectively. Prove that if the triangles ABC and AMN are similar, then $ABCD$ is a square.

SOLUTION. From the similarity of the triangles AMN and ABC, we obtain

$$\frac{AM}{AB} = \frac{AN}{AC} \tag{1}$$

and

$$\angle MAN \equiv \angle BAC \text{ or } \angle BAM \equiv \angle CAN. \tag{2}$$

The relations (1) and (2) imply the similarity of the triangles BAM and CAN. Hence, we obtain the proportions:

$$\frac{AN}{AN} = \frac{AB}{AC} = \frac{BM}{CN} \tag{3}$$

and $\angle ABM \equiv \angle ACN$. The last equality implyies that $ABCD$ is a rectangle.

To conclude the proof, notice that $BM = \frac{1}{4}BD = \frac{1}{4}AC$ and $CN = \frac{1}{2}AB$. Hence the last equality in (3) becomes $\frac{AB}{AC} = \frac{AC}{2AB}$, that is $2AB^2 = AC^2 = AB^2 + BC^2$, which proves that $ABCD$ is a square.

Second JBMO selection examination

PROBLEM 5. A square of side 1 is decomposed into 9 equal squares of sides $\frac{1}{3}$ and the one in the center is painted black. The remaining eight squares are analogously divided into nine squares each, and squares in the centres are painted in black.

Prove that after 1000 steps the total area of the black region exceeds 0.999.

SOLUTION. The first step give rise to one black square of area $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$. After the second step we obtain eight more squares of side $\frac{1}{9}$, the black region increasing thus by $\frac{8}{9^2}$. In the same manner, the third step increases the black area by $8^2 = 64$ black squares, each of area $\frac{1}{27}$, that is at this stage the black area becomes

$$\frac{1}{9} + \frac{8}{9^2} + \frac{8^2}{9^3}.$$

We conclude that after 1000 steps, the area of the black region is

$$\frac{1}{9}+\frac{8}{9^2}+\frac{8^2}{9^3}+\cdots+\frac{8^{999}}{9^{1000}}=\frac{1}{9}\left(1+\frac{8}{9}+\left(\frac{8}{9}\right)^2+\cdots+\left(\frac{8}{9}\right)^{999}\right)$$
$$=\frac{1}{9}\cdot\frac{1-\left(\frac{8}{9}\right)^{1000}}{1-\frac{8}{9}}=1-\left(\frac{8}{9}\right)^{1000}.$$

It is left to prove that the last number is greater than 0.001. This easy follows by using a binomial expansion evaluation, that is

$$\left(\frac{9}{8}\right)^{1000}=\left(1+\frac{1}{8}\right)^{1000}>\binom{1000}{2}\cdot\frac{1}{8^2}=\frac{1000\cdot 999}{2\cdot 64}>1000,$$

the proof being complete.

PROBLEM 6. Find all positive integers a, b, c, d such that

$$a+b+c+d-3=ab+cd.$$

SOLUTION. We have $ab+cd=2(a+b+c+d)-6$ or

$$(a-2)(b-2)+(c-2)(d-2)=2.$$

Assuming that a is the smallest number among a, b, c, d, we get $-1\leqslant a-2\leqslant 1$.

If $a-2=1$ then $b-2=c-2=d-2=1$ and $a=b=c=d=3$.

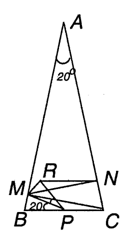
If $a-2=0$, then $c-2=1$ and $d-2=2$ (or $c-2=2$ and $d-2=1$). It follows that $c\cdot d=12, a=2$, that is $b=6$.

If $a-2=-1$, then $a=1$ and $b+c+d-2=b=cd$. Hence $c+d=2$, implying $c=d=1$ and $b=1$.

We conclude that the solutions are $(a,b,c,d)=(3,3,3,3);(1,1,1,1);(2,6,3,4);(6,2,3,4);(2,6,4,3);(6,2,4,3);(3,4,2,6);(3,4,6,2);(4,3,2,6);(4,3,6,2)$.

PROBLEM 7. Let ABC be an isosceles triangle such that $AB=AC$ and $\angle A=20°$. Let M be the foot of the altitude from C and let N be a point on the side AC such that $CN=\frac{1}{2}BC$.

SOLUTION. Let P the midpoint of BC. Since MP is a median in the right-angled triangle MBC, it follows that $PB=MP=PC=CN$.



The point R is considered such that $PCNR$ is a parallelogram (in fact a rhombus). Notice that

$$\angle RPM=\angle RPB-\angle MPB=\angle ACB-(180°-2\angle MBC)=60°,$$

and $RP=MP$, that is MPR is an equilateral triangle. Hence $MR=RP=RN$ and $\angle MRN=\angle MRP+\angle PRN=60°+80°=140°$. Then $\angle RMN=\angle RNM=20°$, $\angle ANM=20°+80°=100°$ and the required angle $\angle AMN$ equals $60°$.

PROBLEM 8. Let $ABCD$ be a unit square. For any interior points M and N such that the line MN does not contain any vertex of the square, denote by $s(M,N)$ the least area of a triangle having vertices in the set $\{A,B,C,D,M,N\}$.

SOLUTION. This is the same problem as Problem 5 for the IMO selection.

Third JBMO selection examination

PROBLEM 9. Let n be an even positive integer and let a, b be two relatively prime positive integers.

Find a and b such that $a+b$ is a divisor of a^n+b^n.

SOLUTION. As n is even, we have

$$a^n-b^n=(a^2-b^2)(a^{n-2}-a^{n-4}b^2+\cdots+b^{n-2}).$$

Since $a+b$ is a divisor of a^2-b^2, it follows that $a+b$ is a divisor of a^n-b^n. In turn, $a+b$ divides $2a^n=(a^n+b^n)+(a^n-b^n)$, and $2b^n=(a^n+b^n)-(a^n-b^n)$. But a and b are coprime numbers, and so g.c.d.$(2a^n,2b^n)=2$. Therefore $a+b$ is a divisor of 2, hence $a=b=1$.

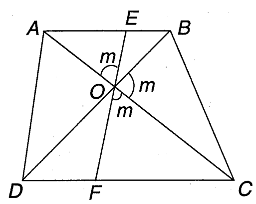
PROBLEM 10. The diagonals AC and BD of a convex quadrilateral $ABCD$ meet at O. Let m be the measure of the acute angle formed by these diagonals. A variable angle xOy of measure m intersects the quadrilateral by a convex quadrilateral of constant area.

Prove that $ABCD$ is a square.

SOLUTION. Consider $\angle AOD=m\leqslant 90°$. As the angles $\angle AOD$ and $\angle BOC$ equal m, we find area(AOD) =area(BOC). It follows

$$\frac{1}{2}AO\cdot DO\cdot\sin m=\frac{1}{2}BO\cdot CO\sin m,$$

that is $\frac{AO}{CO}=\frac{BO}{DO}$. Since $\angle AOB=\angle DOC$, the triangles AOB and DOC are similar and AB is parallel to DC.



Draw line EF that contains O,such that $\angle AOE = \angle COF = m$ and $E \in (AB)$, $F \in (DC)$. The triangles AOE and COF are similar and have the same area, that is they are congruent. It follows that $AO = OC$ and in the same way $BO = OD$. In conclusion $AD \parallel BC$. Moreover area(COF) =area(BOC), and since $ABCD$ is a parallelogram, we find area(BOC) =area(DOC). Hence $D = F$ and $m = \angle COF = \angle DOF = \angle BOC = 90°$. We thus proved that $ABCD$ is a rhombus.

To conclude, consider the bisector lines OM and ON of angles $\angle AOD$ and $\angle DOC$ respectively, where $M \in (AD), N \in (DC)$. It is easy to check that $\angle MON = m = 90°$, whence area(MON) =area(AOD). Thus area(DON) =area(AOM), that is area(AOM) =area(DOM) = $\frac{1}{2}$area(AOD). It follows that OM is a median in the triangle AOD, that is $AO = OD$, which proves that the rhombus $ABCD$ is a square.

PROBLEM 11. A given equilateral triangle of side 10 is divided into 100 equilateral triangles of side 1 by drawing parallel lines to the sides of the original triangle.

Find the number of equilateral triangles, having vertices in the intersection points of parallel lines whose sides lie on the parallel lines.

SOLUTION Let us consider the general case, that is to consider the number a_n of equilateral triangles formed by division in n segments. We shall find a recurence relation.

Consider an equilateral triangle with sides partitioned into $n+1$ equal segments and draw the n parallels to each side of the given triangle. We will count all triangles with at least one vertex on BC; the remaining ones are triangles counted in a_n.

Consider first the triangles that have two vertices on BC. When choosing two division points on BC, one counts exactly one triangle, namely that one obtained by drawing parallels from M_i, N_i to AB, AC respectively. Hence we add $\frac{(n+2)(n+1)}{2}$ new triangles.

Considering triangles with only one vertex on BC, observe that for any point of division, inside the side BC, we count one triangle of side 1, one triangle of side 2, and so on. Hence we add $n + (n-2) + (n-4) + \cdots$ triangles with one side on BC. It follows that we have

$$a_{n+1} = a_n + \frac{(n+2)(n+1)}{2} + n + (n-2) + (n-4) + \cdots.$$

Changing n with $n+1$, we get

$$a_{n+2} = a_{n+1} + \frac{(n+3)(n+2)}{2} + (n+1) + (n-1) + (n-3) = \cdots.$$

Adding up, we obtain

$$a_{n+2} = a_n + \frac{(n+2)(n+1)}{2} + \frac{(n+3)(n+2)}{2} + \frac{(n+1)(n+2)}{2}$$
$$= a_n + \frac{(n+2)(3n+5)}{5}.$$

It follows that

$$a_{10} = a_8 + \frac{10(3\cdot 8+5)}{2} = a_8 + 145 = a_6 + 237 = \cdots = a_0 + 315 = 315.$$

Therefore, the number of triangles is 315.

PROBLEM 12. Prove that for any real numbers a, b, c such that $0 < a, b, c < 1$, the following inequality holds

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

SOLUTION. Observe first that $\sqrt{x} < \sqrt[3]{x}$ for $x \in [0,1]$. Thus, we have $\sqrt{abc} < \sqrt[3]{abc}$ and

$$\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)}.$$

By the AM–GM inequality, we get

$$\sqrt{abc} < \sqrt[3]{abc} \leqslant \frac{a+b+c}{3},$$

and

$$\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)} \leqslant \frac{(1-a)+(1-b)+(1-c)}{3}.$$

Summing up, we obtain

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < \frac{a+1-a+b+1-b+c+1-c}{3} = 1,$$

as desired.

Fourth JBMO selection examination

PROBLEM 13. Let a be an integer. Prove that for any real number x, $x^3 < 3$, both the numbers $\sqrt{3-x^2}$ and $\sqrt[3]{a-x^3}$ cannot be rational.

SOLUTION. Suppose, by way of contradiction, that $u = \sqrt{3-x^2}$ and $v = \sqrt[3]{a-x^3}$ are rational numbers. It follows that $x^2 = 3-u^2$ and $x^3 = a - v^3$, that is $a - v^3 = \pm(3-u^2)\sqrt{3-u^2}$. It follows that $\sqrt{3-u^2} = q$ has to be rational, and $3 = u^2 + q^2$, both u and q being rationals.

Let m, n, p be integers with g.c.d.$(m,n,p) = 1$, such that $u = \frac{m}{p}$ and $v = \frac{n}{p}$. Then $3p^2 = m^2 + n^2$, that is 3 is a divisor of m^2+n^2. It is easy to see that 3 has to be a divisor of both m and n. Furthermore 9 is a divisor of $3p^2$, implying that 3 divides p. Since g.c.d.$(m,n,p) = 1$ we get a contradiction.

PROBLEM 14. The last four digits of a perfect square are equal. Prove that all of them are zeros.

SOLUTION. Denote by n^2 the perfect square and by a the digit that appears in the last four positions. It easily follows that a is one of the numbers $0, 1, 4, 5, 6$ or 9. It follows $n^2 \equiv a \cdot 1111 (\bmod 10^4)$ and consequently $n^2 \equiv a \cdot 1111 (\bmod 16)$.

When $a = 0$ we are done. Suppose that a is 1, 5 or 9. Since $n^2 \equiv 0$ or 1 or 4 $(\bmod 8)$ and $1111 \equiv 7 (\bmod 8)$, we obtain $1 \cdot 1111 \equiv 7 (\bmod 8)$, $5 \cdot 1111 \equiv 3 (\bmod 8)$ and $9 \cdot 1111 \equiv 7 (\bmod 8)$. Thus the congruence $n^2 \equiv a \cdot 1111 (\bmod 16)$ cannot hold.

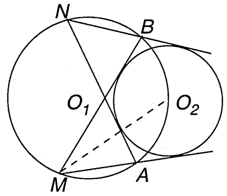
Suppose a is 4 or 6. As $1111 \equiv 7 (\bmod 16)$, $4 \cdot 1111 \equiv 12$ (mod 16) and $6 \cdot 1111 \equiv 10$ (mod 16). We conclude that in this case the congruence $n^2 \equiv a \cdot 1111 (\bmod 16)$ cannot hold, also.

PROBLEM 15. Let $C_1(O_1)$ and $C_2(O_2)$ be two circles such that C_1 passes through O_2. Point M lies on C_1 such that $M \notin O_1O_2$. The tangents from M at C_2 meet again C_1 at A and B.

Prove that the tangents from A and B at C_2 – others than MA and MB – meet at a point located on C_1.

SOLUTION. Since O_2 is at equal distances from the tangents MA and MB, it follows that MO_2 is a bisector line of the angle $\angle AMB$ or of the exterior angle defined by MA and MB.

In the first case we find arcO_2A =arcO_2B. In the second case, using the notations in the figure, we have arcBO_2 =arcM_2+arcAM =arcAO_2 and $O_2B = O_2A$.



Reflecting the figure with respect to the line O_1O_2, the circles C_1 and C_2 remain fixed, A reflects in B and M reflects in N. It is obvious that NA, the reflected of MB is tangent to C_2 and the same is valid for NB. Observe that N is on C_1, proving thus the claim.

PROBLEM 16. Five points are given in the plane that each of 10 triangles they define has area greater than 2. Prove that there exists a triangle of area greater than 3.

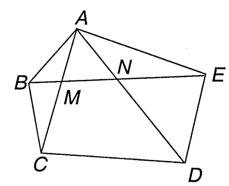
SOLUTION. Denote by A, B, C, D, E the five given points. If the pentagon $ABCDE$ is concave, we can suppose that D is situated inside the triangle ABC or inside the quadrilateral $ABCD$ (see figure).

In first case area(ABC) =area(ABD)+area(DBC)+area$(DAC) > 6 > 3$.

In the second case, D is inside the one of triangles BCE, ACE, ABC or ABD. Suppose, without loss of generality, that D is inside to the triangle BCE. Then

$$\text{area}(BCE) \geqslant \text{area}(BDC) + \text{area}(CDE) > 4 > 3.$$

Consider now the case when $ABCD$ is a convex pentagon. Let M and N be the intersection points of BE with AC and AD respectively.



The following result will be useful.

LEMMA. Let $PQRS$ be a quadrilateral and T a point on the side PQ. Then

$$\text{area}(TRSR) \geqslant \min(\text{area}(PSR), \text{area}(QSR)).$$

The proof consists of simply observing that the distance from T to SR is bounded up and below by the distances from P and Q to SR.

In our case, suppose that $BM \geqslant \frac{1}{3}BE$, which yields $BM \geqslant \frac{1}{2}ME$. Then

$$\begin{aligned}\text{area}(BDE) &= \text{area}(BDM) + \text{area}(MDE) \geqslant \frac{1}{2}\text{area}(MDE) + \text{area}(MDE)\\ &= \frac{3}{2}\text{area}(MDE) \geqslant \frac{3}{2}\min(\text{area}(CDE), \text{area}(ADE)) > \frac{3}{2}\cdot 2 = 3.\end{aligned}$$

The case when $NE \geqslant \frac{1}{3}BE$ is similar. It is left to consider the case when $MN \geqslant \frac{1}{3}BE$. We then have:

$$\text{area}(AMN) \geqslant \frac{1}{3}\text{area}(ABE) > \frac{2}{3},$$

$$\text{area}(MND) \geqslant \frac{1}{3}\text{area}(BED) > \frac{2}{3}$$

and

$$\text{area}(MCD) \geqslant \min(\text{area}(BCD), \text{area}(ECD)) > 2.$$

Summing up, we conclude

$$\text{area}(ACD) > 2 + \frac{2}{3} + \frac{2}{3} > 3,$$

and the proof is complete.

Fifth JBMO selection examination

PROBLEM 17. Let $m, n > 1$ be integer numbers. Solve in positive integers

$$x^n + y^n = 2^m.$$

SOLUTION. Let $d = \text{g.c.d.}(x, y)$ and $x = da$, $y = db$, where $(a, b) = 1$. It is easy to see that a and b are both odd numbers and $a^n + b^n = 2^k$ for some integer k.

Suppose that n is even. As $a^2 \equiv b^2 \equiv 1$ modulo 8 we have also $a^n \equiv b^n \equiv 1$ modulo 8. As $2^k = a^n + b^n \equiv 2(\text{mod}8)$, we conclude $k = 1$ and $a = b = 1$, thus $x = y = d$.

The equation becomes $x^n = 2^{m-1}$. It has an integer solution if and only if n is a divisor of $m - 1$ and $x = y = 2^{\frac{m-1}{n}}$.

Consider the case when n is odd. From the decomposition $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \cdots + b^{n-1})$, we easily get $a + b = 2^k = a^n + b^n$. In this case $a = b = 1$, and the proof goes on the lines of the previous case.

To conclude, the given equation has solutions if and only if $\frac{m-1}{n}$ is an integer and in that case $x = y = 2^p$.

PROBLEM 18. We are given n circles which have the same center. Two lines D_1, D_2 are concurent in P, a point inside all circles. The rays determined by P on the line D_1 meet the circles in points $A_1, A_2, \ldots, A_n$ and $A'_1, A'_2, \ldots, A'_n$ respectively and the rays on D_2 meet the circles at points $B_1, B_2, \ldots, B_n$ and $B'_1, B'_2 \ldots, B'_n$ (points with the same indices lie on the same circle).

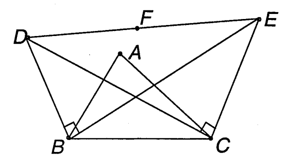
Prove that if the arcs A_1B_1 and A_2B_2 are equal then the arcs A_iB_i and $A'_iB'_i$ are equal, for all $i = 1, 2, \ldots n$.

SOLUTION. Let O be the common center of the n circles and $\alpha = \text{arc}A_1B_1 = \text{arc}A_2B_2$ (the considered arcs are oriented). Rotate the figure about center O by angle α such that A_1, A_2 become B_1, B_2 respectively. The above rotation R maps lines into lines, that is $R(D_1) = D_2$, since $D_1 = A_1A_2$ and $D_2 = B_1B_2$. Moreover, a point M of a circle C_i, with center O, remains on the same circle after rotation. Because $R(A_i)$ lies on D_2 and on C_i, we get that $R(A_i) = B_i$, that is $\text{arc}A_iB_i = \alpha$. In the same way we get $R(A'_i) = B'_i$ and $\text{arc}A'_iB'_i = \alpha$. This concludes the proof.

PROBLEM 19. Let ABC be a triangle and $a = BC$, $b = CA$ and $c = AB$ be the lengths of its sides. Points D and E lie in the same halfplane determined by BC as A. Suppose that $DB = c$, $CE = b$ and that the area of $DECB$ is maximal. Let F be the midpoint of DE and let $FB = x$.

Prove that $FC = x$ and $4x^3 = (a^2 + b^2 + c^2)x + abc$.

SOLUTION. Let $DECB$ be the quadrilateral of maximal area. It is easy to prove that $\angle DBE = \angle DCE = 90°$.



It follows $FB = FC = \frac{DE}{2} = x$ and that the quadrilateral $DBCE$ is cyclic. By Ptolemey's theorem we have $DC \cdot BE = BC \cdot DE + DB \cdot CE$. Squaring, we get

$$(4x^2 - b^2)(4x^2 - c^2) = (2ax + bc)^2,$$

that is

$$4x^4 - (b^2 + c^2)x^2 + b^2c^2 = 4a^2x^2 + 4abcx + b^2c^2.$$

From this we obtain $4x^3 = (a^2 + b^2 + c^2)x + abc$, as desired.

PROBLEM 20. Let p, q be two distinct primes. Prove that there are positive integers a, b such that the arithmetic mean of all positive divisors of the number $n = p^aq^b$ is an integer.

SOLUTION. The sum of all divisors of n is given by the formula

$$(1 + p + p^2 + \cdots + p^a)(1 + q + q^2 + \cdots + q^b),$$

as it can be easily seen by expanding the brackets. The number n has $(a+1)(b+1)$ positive divisors and their arithmetic mean is

$$m = \frac{(1 + p + p^2 + \cdots + p^a)(1 + q + q^2 + \cdots + q^b)}{(a + 1)(b + 1)}.$$

If p and q are both odd numbers, we can take $a = p$ and $b = q$ and it is easy to see that m is an integer.

If $p = 2$ and q is odd, one can choose again $b = q$ and consider $a + 1 = 1 + q^2 + \cdots + q^{q-1}$. Then $m = 1 + 2 + 2^2 + \cdots + 2^a$, and it is an integer. For p odd and $q = 2$, we choose $a = p$ and $b = p^2 + \cdots + p^{p-1}$, concluding the proof.

Part IV: SIXTH REGIONAL CONTEST "ALEXANDRU PAPIU ILARIAN"

Târgu Mureş, October 27, 2001

PROPOSED PROBLEMS

9th GRADE

PROBLEM 1. Find the least positive integer n such that 3^{2001} is a divisor of $(n+1)(n+2)\cdots 3n$.

Ion Cheşcă

PROBLEM 2. Find the first 2001 digits of the number

$$\sqrt{0.\underbrace{11\cdots 1}_{2001 \text{ times}}}.$$

Dorin Popovici

PROBLEM 3. Let A be a 2001 point set in the plane. Show that there exists a circle which passes through exactly one point of A and contains 1000 points of A in its interior.

Marian Andronache and Ion Savu

PROBLEM 4. Let G be the centroid of the triangle ABC. The line d through G intersects the sides BC, CA, AB in the points A_1, B_1, C_1 respectively. Let L be an interior point of the triangle which is also contained in d. Show that

$$\frac{LA_1}{A_1G} + \frac{LB_1}{B_1G} + \frac{LC_1}{C_1G} = 3.$$

Marian Dincă

10th GRADE

PROBLEM 1. Let a, b be real numbers such that $a < b$. Show that the interval (a, b) contains infinitely many irrational numbers x such that x^3 is a rational number.

Valentin Matrosenco and Marcel Chiriţă

PROBLEM 2. Show that for any distinct positive integers $a_1, a_2, \ldots, a_n$, the following inequality holds:

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geqslant \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

Laurenţiu Panaitopol

PROBLEM 3. We are given $n+1$ vectors in the plane which have the lengths $1, 2, \ldots, 2^n$. Show that the sum of any subset of these vectors is a vector of length at least 1.

Barbu Berceanu

PROBLEM 4. We are given $2n$ distinct points in a plane. Show that there exist n disjoint segments which connect n pairs of these points.

Mathematical Olympiad of Republic of Moldova

11th GRADE

PROBLEM 1. Show that no four points on the graph of a convex real function are the vertices of a parallelogram.

Romeo Ilie

PROBLEM 2. Let $a_1, a_2, \ldots a_n, \ldots$ be an arithmetic progression of positive integers. Show that for any integer k, the sequence

$$\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}, \ldots$$

contains k numbers in arithmetic progression.

Marian Andronache and Ion Savu

PROBLEM 3. Let $a_0, a_1, \ldots, a_n$, be real numbers such that the equation

$$a_0 + a_1 \cos x + \cdots + a_n \cos nx = 0$$

has $2n+1$ distinct solutions in the interval $[0, 2\pi]$. Show that

$$a_0 = a_1 = \cdots = a_n = 0.$$

Sorin Rădulescu and Ion Savu

PROBLEM 4. Let $n \geqslant 4$ be an integer number. We are given a pyramid $SA_1A_2\ldots A_n$, whose base is the convex polygon $A_1A_2\ldots A_n$. A plane intersect the edges $SA_1, SA_2, \ldots, SA_n$ in the points $B_1, B_2, \ldots, B_n$, respectively. Show that if the polygons $A_1A_2\ldots A_n$ and $B_1B_2\ldots B_n$ are similar, then the planes containing them, are parallel.

Laurenţiu Panaitopol

12th GRADE

PROBLEM 1. a) Show that for any positive integer n, the following inequality holds

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} > \ln(n+1).$$

b) Let $(x_n)_{n\geqslant 1}$ be a sequence of positive numbers such that

$$x_1 + x_2 + \cdots + x_n \leqslant n^2,$$

for all $n \geqslant 1$. Show that

$$\lim_{n\to\infty}\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) = \infty.$$

Mihai Piticari

PROBLEM 2. Let $f, g : \mathbf{R} \to \mathbf{R}$ be real functions.
a) Show that if f, g have the intermediate value property, then so $g \circ f$.
b) Show that if g has the intemediate value property and

$$|f(x) - f(y)| \leqslant |g(x) - g(y)|,$$

for all real numbers x, y, then f has the intermediate value property as well.

Sórin Rădulescu and Marius Rădulescu

PROBLEM 3. Show that any $n \times n$ real matrix can be written as a difference of two real matrices of the same size and which have negative determinants.

George Stoianovici

PROBLEM 4. Let n be a positive integer. Show that there exists a matrix with real entries $n \times n$ and such that all its square submatrices have positive determinants.

Ion Savu

SOLUTIONS

9th GRADE

PROBLEM 1. Find the least positive integer n such that 3^{2001} is a divisor of $(n+1)(n+2)\cdots 3n$.

SOLUTION. We look for the exponent of 3 in the prime decomposition of the number $a_n = (n+1)(n+2)\cdots(3n)$. An elementary remark gives us $a_n = 3^n \cdot 1 \cdot 2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdots (3n-2)(3n-1)$. Hence the exponent of 3 in a_n is n. It follows that the required number is 2001.

PROBLEM 2. Find the first 2001 digits of the number

$$\sqrt{0.\underbrace{11\cdots1}_{2001\text{ times}}}.$$

Dorin Popovici

SOLUTION. Let $a = 0.11\ldots1$ be the given number. We have $9a = 0.99\ldots9 = 1 - \frac{1}{10^{2001}} < 1$. For any positive real number x such that $0 < x < 1$, one has

$x < \sqrt{x} < 1$. Then, $9a < \sqrt{9a} = 0.99\ldots9\ldots$. We also have $\sqrt{9a} = 3\sqrt{a}$. Therefore $\sqrt{a} = 0.33\ldots3\ldots$. The conclusion is: first 2001 decimals are 3's.

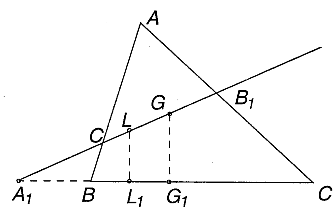
PROBLEM 3. Let A be a 2001 point set in the plane. Show that there exists a circle which passes through exactly one point of A and contains 1000 points of A in its interior.

SOLUTION. Let $A = \{A_1, A_2, \ldots, A_{2001}\}$ be the point set. There are $\frac{2001\cdot 2000}{2}$ segments which have their edges in the set A. We have $\frac{2001\cdot 2000}{2}$ bisector lines of these segments. Take an arbitrary point O in the plane which is not on any of these bisectors. The distances $d_i = OA_i$ are all distinct and we may put them on an increasing sequence. Consider a circle with center O and radius d_{1001}. It satisfies the required conditions.

PROBLEM 4. Let G be the centroid of the triangle ABC. The line d through G intersects the sides BC, CA, AB in the points A_1, B_1, C_1 respectively. Let L be an interior point of the triangle which is also contained in d. Show that

$$\frac{LA_1}{A_1G} + \frac{LB_1}{B_1G} + \frac{LC_1}{C_1G} = 3.$$

SOLUTION. For any triangle XYZ of the plane, let us denote by S_{XYZ} its area and let $S = S_{ABC}$. Consider the points L_1, G_1, the feet of the perpendiculars from L and G respectively, on BC.



Then

$$\frac{A_1L}{A_1G} = \frac{LL_1}{GG_1} + \frac{S_{LBC}}{S_{GBC}}.$$

Because $S_{GBC} = \frac{1}{3}S$, we obtain

$$\frac{A_1L}{A_1G} = \frac{3S_{LBC}}{S}.$$

In a similar way $\frac{B_1L}{B_1G} = \frac{3S_{LBC}}{S}$ and $\frac{C_1L}{C_1G} = \frac{3S_{LAB}}{S}$. Summing up and using $S = S_{LAB} + S_{LBC} + S_{LCA}$ we get the result.

10th GRADE

PROBLEM 1. Let a, b be real numbers such that $a < b$. Show that the interval (a, b) contains infinitely many irrational numbers x such that x^3 is a rational number.

SOLUTION. If an integer number a has the form $a = 4n + 2$, then it is not a perfect cube. It follows that for any integer m, the number $x = \frac{\sqrt[3]{4n+2}}{m}$ is irrational and x^3 is rational.

It suffices to prove that for any c, d, $c < d$, we can find infinitely many integers m, n, such that

$$c < \frac{4n+2}{m^3} < d,$$

which can be easily proved.

PROBLEM 2. Show that for any distinct positive integers $a_1, a_2, \ldots, a_n$, the following inequality holds:

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geqslant \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

SOLUTION. We introduce the quadratic function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^2 - \frac{2n+1}{3}x$. It is decreasing on the interval $[0, \frac{2n+1}{6})$ and increasing on the interval $(\frac{2n+1}{6}, +\infty)$. We have

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geqslant 0.$$

It is easy to see that $f(1) + f(2) + \cdots + f(n) = 0$. For any x, $x \geqslant n + 1$, one also has $f(x) \geqslant f(k)$ for $k = 1, 2, \ldots, n$. Therefore if we change the set $\{1, 2, \ldots, n\}$ with another set of distinct positice integers $\{a_1, a_2, \ldots, a_n\}$, we have

$$\sum_{i=1}^{n} f(a_i) \geqslant \sum_{i=1}^{n} f(i) \geqslant 0.$$

This ends the proof.

PROBLEM 3. We are given $n+1$ vectors in the plane which have the lengths $1, 2, \ldots, 2^n$. Show that the sum of any subset of these vectors is a vector of length at least 1.

SOLUTION. Let $\{v_{i_1}, \ldots, v_{i_n}\}$ a subset of vectors of the given set. We may assume that $i_1 < \cdots < i_n$. Then

$$|v_{i_1} + \cdots + v_{i_k}| \geqslant |v_{i_k}| - |v_{i_1} + \cdots + v_{i_{k-1}}| \geqslant |v_{i_k}| - |v_{i_1}| - \cdots - |v_{i_{k-1}}|$$
$$\geqslant 2^{i_k} - 2^{i_{k-1}} - \cdots 2^{i_1} \geqslant 1.$$

PROBLEM 4. We are given $2n$ distinct points in a plane. Show that there exist n disjoint segments which connect n pairs of these points.

SOLUTION. there are $n(2n-1)$ segments which connect pairs of given points. There are $\binom{n(2n-1)}{n}$ possibilities to choose a familly of n segments from the set of all segments. For each familly of n segments we compute the sum of their length and choose a familly for which the sum is least. We will prove that this familly satisfies the required condition.

If by contradiction, two segments, say AB and CD have a common point O, then $AC + BD < AO + OC + DO + OB = AB + CD$. Therefore, when changing AB, CD with AC and BD the sum of length decreases. This contradicts the minimality.

11th GRADE

PROBLEM 1. Show that no four points on the graph of a convex real function are the vertices of a parallelogram.

SOLUTION. Assume by contradiction, that there exist real numbers $a < b < c < d$ such that the points $A(a, f(a)), B(b, f(b)), C(c, f(c)), D(d, f(d))$ are the vertices of a parallelogram.

We have $a + d = b + c$ and $f(a) + f(d) = f(b) + f(c)$. Suppose $a < b < c < d$. This implies the existence of $\lambda, \mu \in (0, 1)$ such that $b = \lambda a + (1 - \lambda)d$ and $c = \mu a + (1 - \mu)d$. The convexity implies $f(b) < \lambda f(a) + (1 - \lambda)f(d)$ and $f(c) < \mu f(a) + (1 - \mu)f(d)$. Summing up we get $\lambda + \mu = 1$ and $f(b) + f(c) < (\lambda + \mu)f(a) + (2 - \lambda - \mu)f(d)$. This gives $f(b) + f(c) < f(a) + f(d)$, which is a contradiction.

PROBLEM 2. Let $a_1, a_2, \ldots a_n, \ldots$ be an arithmetic progression of positive integers. Show that for any integer k, there are k numbers in arithmetic progression, in the sequence

$$\frac{1}{a_1}, \frac{1}{a_2} \cdots, \frac{1}{a_n}, \cdots$$

SOLUTION. Suppose $a_n = a_1 + (n-1)r$, $r > 0$, gives the progression. For $p \geqslant 3$, the number $N = a_1(1 + r)(1 + 2r) \cdots (1 + (p-1)r)$ is a member of the progression, as being of the form $a_1 + kr$.

Remark that the numbers $b_1 = \frac{1}{n}, b_2 = \frac{1+r}{N}, \ldots, b_p = \frac{1+(p-1)r}{N}$, all belong to the set $\frac{1}{a_n}$, and are in arithmetical progression.

PROBLEM 3. Let $a_0, a_1, \ldots, a_n$, be real numbers such that the equation

$$a_0 + a_1 \cos x + \cdots + a_n \cos nx = 0$$

has $2n + 1$ distinct solutions in the interval $[0, 2\pi]$. Show that

$$a_0 = a_1 = \cdots = a_n = 0.$$

SOLUTION. Consider the complex number $z = \cos x + \mathrm{i} \sin x$. We get $\cos kx = \frac{1}{2}(z^k + \frac{1}{z^k})$. The given equation transforms in a polynomial equation of degree $2n$ with $2n + 1$ distinct roots. The polynomial coefficient's are thus all zero. This imlply that all the a_i's are zero.

PROBLEM 4. Let $n \geqslant 4$ be an integer number. We are given a pyramid $SA_1A_2 \ldots A_n$, whose base is the convex polygon $A_1A_2 \ldots A_n$. A plane intersect the edges $SA_1, SA_2, \ldots, SA_n$ in the points $B_1, B_2, \ldots, B_n$, respectively. Show that if the polygons $A_1A_2\dots A_n$ and $B_1B_2\dots B_n$ are similar, then the planes containing them, are parallel.

SOLUTION. Denote by V_X the volume of any solid X. We then have, by elementary results

$$\frac{V_{SB_1B_2B_3}}{V_{SA_1A_2A_3}} = \frac{SB_1\cdot SB_2\cdot SB_3}{SA_2\cdot SA_2\cdot SA_3} = \lambda\frac{h}{H},$$

where λ is the ratio of similitude of the given polygons, h and H respectively, are the altitudes of the two pyramids.

By similar calculations we finaly get that

$$\frac{SB_1}{SA_1} = \frac{SB_4}{SA_4} = \dots = \frac{SB_n}{SA_n},$$

which implies that the two planes are parallel.

12th GRADE

PROBLEM 1. a) Show that for any positive integer n, the following inequality holds

$$\frac{1}{1}+\frac{1}{2}+\dots+\frac{1}{n} > \ln(n+1).$$

b) Let $(x_n)_{n\geqslant 1}$ be a sequence of positive numbers such that

$$x_1+x_2+\dots+x_n \leqslant n^2,$$

for all $n\geqslant 1$. Show that

$$\lim_{n\to\infty}\left(\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_n}\right)=\infty.$$

SOLUTION. a) The proof is a standard use of Lagrange's formula for the function $f(x)=\ln x$.

b) Consider $y_n = \frac{1}{x_1}+\dots+\frac{1}{x_n}$. It is an increasing sequence. It will suffice to prove that the limit is not finite.

We have

$$y_{2n}-y_n \geqslant \frac{n^2}{x_{n+1}+\dots+x_{2n}},$$

by Cauchy-Schwarz inequality, and because

$$\frac{n^2}{x_{n+1}+\dots+x_{2n}} > \frac{n^2}{x_1+x_2+\dots+x_{2n}} \geqslant \frac{n^2}{4n^2} = \frac{1}{4},$$

we get the result.

PROBLEM 2. Let $f,g:\mathbf{R}\to\mathbf{R}$ be real functions.

a) Show that if f,g have the intermediate value property, then so $g\circ f$.

b) Show that if g has the intemediate value property and

$$|f(x)-f(y)| \leqslant |g(x)-g(y)|,$$

for all real numbers x,y, then f has the intermediate value property as well.

SOLUTION. a) It is a standard use of the definition.

b) Using the given condition, one can define a function $h:\mathrm{Im}(g)\to\mathrm{Im}(f)$ by $h(g(x))=f(x)$. The function h is easily seen to be continuous by the given condition. As $\mathrm{Im}(g)$ is an interval, the conclusion follows from a).

PROBLEM 3. Show that any $n\times n$ real matrix can be written as a difference of two real matrices of the same size and which have negative determinants.

SOLUTION. Write given matrix as the difference of two triangular matrices both having positive products of the elements on the main dyagonal.

PROBLEM 4. Let n be a positive integer. Show that there exists a matrix with real entries of size $n\times n$ such that all its square submatrices have positive determinants.

SOLUTION. We shall construct the matrix by induction. For $n=2$ the matrix $A=\begin{pmatrix}1&1\\1&2\end{pmatrix}$ satisfies the condition.

Suppose A_n was constructed for n, and consider the matrix A_{n+1} of the following form

$$A_{n+1}=\begin{pmatrix} & & & & 1\\ & & & & x^3\\ & & A_n & & x^{3^2}\\ & & & & x^{3^3}\\ & & & & \vdots\\ 1 & x^3 & x^{3^2} & x^{3^3} \;\dots & x^{3^n}\end{pmatrix},$$

where $x>0$ is large enough.

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